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Significance of Initial Interpolation in Band-Limited Signal Interpolation

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Abstract—In this correspondence, we present an improved version of the Papoulis algorithm for band-limited signal interpolation. This algorithm uses the concept of initial interpolation. The justification for initial interpolation is developed only through experimental studies. We show that the performance of the interpolation scheme depends on the number and distribution of the known data samples.

I. Introduction

The problem of estimating the values of the missing samples of a signal from partial data arises in several practical situations. An algorithm for interpolation which takes the spectral information into account was presented by Naidu and Paramasiviah [1]. This algorithm, however, did not converge fast in many cases because of the assumption that the initial estimates for the missing data values are set to zero. In this correspondence, we first show the need for initial interpolation of data, and then we present an initial interpolation scheme. We also investigate the performance of our algorithm with and without initial interpolation for different data distributions.

II. Initial Interpolation

In the band-limited interpolation problem, the general tendency appears to be to set all the missing samples to zero. It is not obvious why zero should be an automatic choice. Ideally we would like an estimate which is not only consistent with all the available information, but which is also maximally noncommittal with regard to the unavailable information. We develop an empirical approach to initial interpolation based on some heuristic arguments. We pro-

pose that the interpolated value be made of two components—one dependent on the given data, and the other a noise component, that is added to increase the uncertainty.

A. Data-Dependent Component

We have examined some of the standard (linear and polynomial) interpolation techniques and also a relaxation procedure which was proposed as a constraint propagation algorithm [2]. We found that the relaxation procedure performs better compared to the standard interpolation techniques. This procedure is described by the equation

$$x_i(n) = C_i x_{i-1}(n) + \frac{1 - C_i}{2} \left[x_i(n - 1) + x_{i-1}(n + 1) \right] \quad (1)$$

where

$x_i(n) =$ interpolated value of the nth instant at the end of the ith iteration; wherever appropriate, known values are used for $x_i(n - 1)$ and $x_{i-1}(n + 1)$;

$C_i = i/N =$ confidence factor;

$N =$ number of iterations for the initial interpolation.

The relaxation procedure exploits the constraints supplied by the known sample values by propagating these to even farther away unknown samples. Thus, the available information is more effectively used through this procedure than by the standard interpolation techniques.

B. Noise Component

The "dependency" relation between samples of a signal becomes weak when the known samples are far apart. A noise component $y(n)$ was chosen to add a randomizing feature to the unknown sample values.

$$y(n) = \frac{4\bar{y}(n_2 - n)(n - n_1)}{(n_2 - n_1)} \quad (2)$$

where

n_1 and n_2 are the instants of the two consecutive known samples

$y(n) =$ noise value at the sample instant n

$\bar{y} =$ a random number uniformly distributed between maximum and minimum values of the known samples.

The two components are combined to given the initially interpolated signal as

$$z(n) = C_1 x(n) + C_2 y(n) \quad (3)$$

where C_1 and C_2 are usually set to 1. But for a given data, some optimal values for C_1 and C_2 can be obtained experimentally.

C. An Example

We illustrate our iterative signal interpolation method through simulated examples. We consider a signal of the type

$$s(n) = a_1 \sin (2\pi f_1 n T_S + \theta_1) + a_2 \sin (2\pi f_2 n T_S + \theta_2) \quad (4)$$

where $T_S = 1/256$, $a_1 = 1.25$, $a_2 = 1.50$, $\theta_1 = \pi/3$, $\theta_2 = \pi/2$.

The data were generated by multiplying the complete signal $s(n)$ with a random sequence of ones and zeros. Fig. 1(a) shows the signal for $f_1 = 10$ and $f_2 = 12$, and Fig. 1(b) shows its spectrum (log magnitude). Fig. 1(c) shows a data distribution obtained by selecting 10 percent of the signal samples. Fig. 1(e) shows the interpolated values. In this, 20 iterations of the relaxation procedure were used. The spectra of the data distribution and that of the interpolated sequence are shown in Fig. 1(d) and (f), respectively. We notice that the signal peaks stand out much better in Fig. 1(f) than in Fig. 1(d). The contribution of the noise component is to
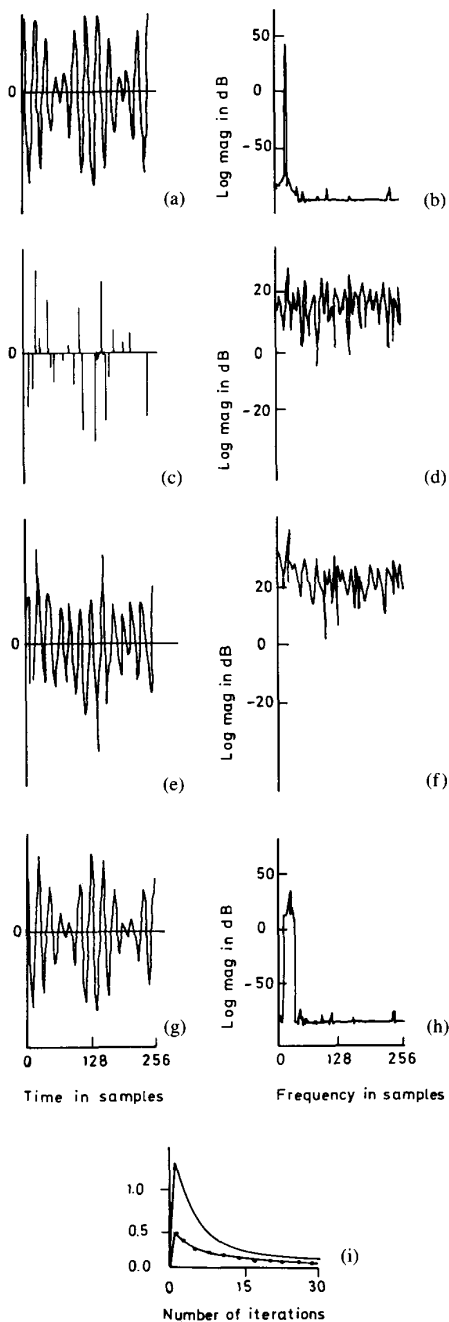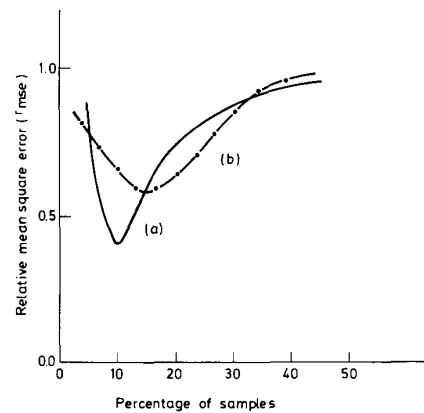
Fig. 2. Illustration of the performance of the initial interpolation method for different percentages of known samples in the data. The ratio (r_{mse}) of mean-square errors with and without initial interpolation after 5 interations is plotted for this illustration; (a) for a low-frequency signal; (b) for a high-frequency signal.

Fig. 1. Illustration of the proposed algorithm for interpolation of a low-frequency signal. (a) Original time domain signal. (b) Spectrum of the original signal. (c) Partial data consisting of 10 percent of the original signal. (d) Spectrum of the partial data. (e) Initial interpolated signal using relaxation and noise. (f) Spectrum of the initial interpolated signal. (g) The reconstructed signal. (h) Spectrum of the reconstructed signal. (i) Mean square error with (dotted curve) and without initial interpolation.

reduce the effect of low-frequency bias produced by the relaxation procedure.

To extend the interpolation scheme to handle high-frequency signals as well, we can modify the procedure as follows. Given the knowledge of the band location, we can find the center frequency f_o. The data can then be used to demodulate the data by multiplying

it with cos $\omega_o n$. This demodulation translates the data spectrum to a low-frequency band. Now the relaxation technique can be applied. The original spectral band can then be recovered through remodulation by multiplying the interpolated data with cos $\omega_o n$.

To these initially interpolated data, the Papoulis algorithm as given in [1] is applied. In order to evaluate the performance of our iterative procedure, we adopt the mean-square error of the reconstructed samples from the original samples as the error criterion. The convergence performance can be seen by the error curves of Fig. 1(i), where the mean-square errors with and without the initial interpolation procedure are plotted against the number of iterations. For a given number of iterations, it can be seen that the error is significantly smaller with initial interpolation than without.

III. PERFORMANCE EVALUATION

A. Effect of Sample Distribution on Reconstruction

A study was conducted to investigate whether the way in which the data samples are distributed affects the reconstruction. Assuming 10 percent for the known signal samples, the error curves for different sample distributions were obtained. In most cases, the asymptotic value of the error is lower with initial interpolation than without. In the extreme case of a cluster of known samples, the proposed initial interpolation does not work, as this corresponds to extrapolation.

The performance for different percentages of samples in the data is illustrated in Fig. 2. It shows the plot of the ratio of the mean-square error with and without initial interpolation versus the percentage of randomly distributed samples in the data. Two curves are shown, one for a low-frequency signal ($f_1 = 10, f_2 = 12$), and the other for a high-frequency ($f_1 = 140, f_2 = 150$) signal. The curves show that the performance is best when the percentage of known samples is within a certain range. When the percentage of the known samples is too low (<5 percent) or too high (>50 percent), the effect of initial interpolation is not significant. This is understandable because, in the former case, the known samples are too few for any interpolation scheme to work properly. In the latter case, the known samples are sufficient to indicate the spectral peaks clearly, and hence no further spectral peak enhancement would be truly necessary.

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