Significance of Group Delay Functions in Signal Reconstruction from Spectral Magnitude or Phase

B. YEGNANARAYANA, D. K. SAIKIA, AND T. R. KRISHNAN

Abstract—**In this paper we discuss the problem of signal reconstruction from spectral magnitude or phase using group delay functions. We define two separate group delay functions for a signal, one is derived from the magnitude and the other from the phase of the Fourier transform of the signal. The group delay functions offer insight into the problem of signal reconstruction and suggest methods for reconstructing signals from partial information such as spectral magnitude or phase. We examine the problem of signal reconstruction from spectral magnitude or phase on the basis of these two group delay functions and derive the conditions for signal reconstruction. Based on existing iterative and noniterative algorithms for signal reconstruction, we propose new algorithms for some special classes of signals. The algorithms are illustrated with several examples. Our study shows that the relative importance of spectral magnitude and phase depends on the nature of signals. Speech signals are used to illustrate the importance of spectral magnitude and picture signals are used to illustrate the importance of phase in signal reconstruction problems. Using the group delay functions, we explain the convergence behavior of the existing iterative algorithms for signal reconstruction.**

I. Introduction

In general the Fourier transform representation of a signal is complete only when both the spectral magnitude and phase are specified. However, there are certain conditions under which a signal can be completely specified (to within a time shift) by the magnitude of its Fourier transform, or (to within a scale factor) by the phase of its Fourier transform [1], [2]. The objective of this paper is to illustrate these conditions for signal reconstruction through group delay functions of a signal. Viewing the problem of signal reconstruction from this angle suggests algorithms for implementing the reconstruction as well. The key ideas used in our development are i) the relation between the cepstral coefficients and the group delay functions of a signal, ii) group delay functions derived separately from the spectral magnitude and phase of a signal, and iii) the relation between these group delay functions for certain types of signals.

Iterative techniques have been developed recently for reconstruction of a signal from its spectral magnitude or phase when the signal satisfies certain conditions [1], [2]. The unknown phase (or magnitude) is retrieved gradually by imposing appropriate conditions in the time domain, and the known magnitude (or phase) in the frequency domain in successive iterations. This technique has been extended to the reconstruction of a two-dimensional picture signal from its phase [1], [3]. The results of these and other studies along these lines have been used to justify the importance of phase information in signals [4].

In the above studies, the conditions for signal reconstruction were derived in terms of the z transform of the signal. In this paper, these conditions are rederived using group delay functions obtained separately from spectral magnitude and phase functions. The duality in the conditions for signal reconstruction from spectral magnitude and phase is brought forth using these group delay functions which play a unifying role for the two reconstruction problems. We also develop noniterative algorithms and a combination of iterative and noniterative algorithms for signal reconstruction under a variety of conditions. We show that spectral magnitude and phase have different roles to play and that their relative importance in signal reconstruction depends on the nature of the signal. For some signals it is the magnitude that is important for signal reconstruction, and for some other signals it is the phase that is important. For some special types of signals we require partial information about both the magnitude and phase, and for a general signal, of course, we require both the spectral magnitude and phase for reconstructing the signal.

The organization of the paper is as follows. In Section II we state the problem of signal reconstruction. With a view to make this paper reasonably self contained, we introduce in this section the necessary terminology in signal processing relevant to this paper. In Section III the conditions for signal reconstruction are discussed in terms of group delay functions. The algorithms for signal reconstruction are described in Section IV. Some applications of these algorithms to practical signals and their consequences are discussed in Section V. An explanation of the process of convergence in the existing iterative reconstruction algorithms is discussed in Section VI.

Throughout this paper, the terms "magnitude" and "phase" refer, respectively, to the magnitude and phase of the Fourier transform of the signal. Likewise, the terms "poles" and "zeros" refer to the poles and zeros, respectively, of the z transform of the signal sequence, and the term "unit circle" refers to the unit circle in the z plane.

Manuscript received April 15, 1983; revised January 11, 1984.

B. Yegnanarayana is with the Department of Computer Science and Engineering, Indian Institute of Technology, Madras 600 036, India.

D. K. Saikia is with Hindustan Computers Ltd., New Dehli 19, India.

T. R. Krishnan is with the Department of Computer Science, University of Wisconsin, Madison, WI 53706.

0096-3518/84/0600-0610$01.00 © 1984 IEEE

II. Problem of Signal Reconstruction and Related Terminology

A. The Problem of Signal Reconstruction from Spectral Magnitude or Phase

Let $\{x(n)\}$ be a discrete time signal obtained by sampling a continuous time signal $x(t)$. Further, let $\{x(n)\}$ be of finite duration, i.e., $x(n) = 0$ outside the interval $0 \leqslant n \leqslant N-1$. The process of sampling, truncation, and quantization, performed in order to render the signal amenable for digital processing, involves some loss of information about the signal.

Let $X(z)$ be the z-transform of $\{x(n)\}$. Then

$$X(z) = \sum_{n=0}^{N-1} x(n)\, z^{-n}. \tag{1}$$

Restricting the z transform of the sequence $\{x(n)\}$ to be a rational function yields a compact representation with fewer coefficients, but this results in further loss of information. We can express $X(z)$ in the form

$$X(z) = Az^{-n_0} \frac{N(z)}{D(z)} \tag{2}$$

where A is a real constant, n_0 is an integer, and $D(z)$ and $N(z)$ are polynomials in z^{-1}. The roots of the polynomials $D(z)$ and $N(z)$ are respectively the poles and zeros of $X(z)$. The factor z^{-n_0} represents a shift of n_0 samples in the time domain and contributes a linear phase component to the phase function. Thus the pole zero locations of $X(z)$ in the z plane specify $\{x(n)\}$ to within a time shift and a scale factor.

The frequency response $X(e^{j\omega})$ of $\{x(n)\}$ is its z transform evaluated on the unit circle. That is

$$X(e^{j\omega}) = X(z)\big|_{z=e^{j\omega}} \tag{3}$$

and it can be represented by its magnitude and phase as

$$X(e^{j\omega}) = |X(\omega)|\, e^{j\theta(\omega)} \tag{4}$$

where $|X(\omega)|$ represents the magnitude of $X(e^{j\omega})$ and $\theta(\omega)$ is its argument. In digital processing we deal only with samples of the frequency response specified at discrete frequencies. That is

$$X(k) = X(e^{j\omega})\big|_{\omega = 2\pi k/M}, \qquad k = 0, 1, \cdots, M-1. \tag{5}$$

Sampling $X(e^{j\omega})$ results in some loss of information due to aliasing in the time domain.

In the following discussion we assume that sampling in the time and frequency domains is done at sufficiently close intervals so that $\{x(n)\}$ can be accurately obtained from $\{X(k)\}$ and vice versa, i.e., there is no aliasing in either domain. Throughout this paper, we use the function $X(\omega)$ and $X(e^{j\omega})$ interchangeably, and we use the function $X(k)$ to refer to the sampled version of $X(\omega)$. Likewise, for any other function of frequency such as $\tau(\omega)$, the notation $\tau(k)$ refers to its sampled version.

The problem in signal reconstruction is the following. Given the samples of the spectral magnitude $|X(k)|$ or the samples of the phase $\theta(k)$ of a signal, one must determine the conditions under which the signal $\{x(n)\}$ can be reconstructed and obtain algorithms for reconstructing the signal.

B. Terminology

1) Causality: A signal $\{x(n)\}$ is said to be causal if $x(n) = 0$ for all negative values of n.

2) *Unwrapped Phase Function:* The frequency response of a signal $\{x(n)\}$ can be represented as

$$X(\omega) = |X(\omega)|\, e^{j[\theta(\omega) + 2\pi\lambda(\omega)]} \tag{6}$$

where $-\pi \leqslant \theta(\omega) \leqslant \pi$, and $\lambda(\omega)$ is an integer such that $[\theta(\omega) + 2\pi\lambda(\omega)]$ is a continuous function of ω. Then $\theta(\omega)$ is called the principal phase value or the wrapped phase function and $[\theta(\omega) + 2\pi\lambda(\omega)]$ is called the unwrapped phase function.

3) Zero Phase Signal: The inverse Fourier transform of $|X(\omega)|$ gives a zero phase signal.

4) All-Pass Signal: The inverse Fourier transform of $e^{j\theta(\omega)}$ gives an all-pass signal.

5) Minimum Phase Signal: In (2), if $n_0 = 0$ and if all the roots of $N(z)$ and $D(z)$ of $X(z)$ are inside the unit circle, then $\{x(n)\}$ is said to be a minimum phase signal.

6) Maximum Phase Signal: For a given n_0 in (2), if all the poles and zeros of $X(z)$ lie outside the unit circle, then $\{x(n)\}$ is said to be a maximum phase signal.

7) Mixed Phase Signal: For a given n_0 in (2), if the poles and zeros of $X(z)$ lie both inside and outside the unit circle, then $\{x(n)\}$ is said to be a mixed phase signal.

8) Minimum Phase Equivalent Signals:

a) From Magnitude: For a given $|X(\omega)|$ of $\{x(n)\}$, there exists a unique phase function $\phi(\omega)$ such that the inverse Fourier transform of $|X(\omega)|\, e^{j\phi(\omega)}$ is a minimum phase signal, say $\{y(n)\}$. The signal $\{y(n)\}$ is called the minimum phase equivalent of $\{x(n)\}$ derived from the magnitude. The poles (zeros) of $\{x(n)\}$ outside the unit circle are reflected as poles (zeros) at conjugate reciprocal locations in $\{y(n)\}$.

b) From Phase: Let $\theta(\omega)$ be the phase function of $\{x(n)\}$ in which the linear phase component is removed. Then there exists a unique magnitude function $|Y(\omega)|$ such that the inverse Fourier transform of $|Y(\omega)|\, e^{j\theta(\omega)}$ is a minimum phase signal, say $\{y(n)\}$. The signal $\{y(n)\}$ is called the minimum phase equivalent of $\{x(n)\}$ obtained from the phase. The poles (zeros) of $\{x(n)\}$ outside the unit circle are reflected as zeros (poles) at conjugate reciprocal locations in $\{y(n)\}$.

C. Relationship Between Spectral Magnitude and Phase of a Minimum (or Maximum) Phase Signal Through Cepstral Coefficients

Let the Fourier transform $V(\omega)$ of a minimum phase signal $\{v(n)\}$ be represented as

$$V(\omega) = |V(\omega)|\, e^{j\theta_v(\omega)} \tag{7}$$

Then we can show that [5]

$$\ln |V(\omega)| = c(o)/2 + \sum_{n=1}^{\infty} c(n) \cos n\omega \tag{8}$$

and the unwrapped phase function

$$\theta(\omega) = \theta_v(\omega) + 2\pi\lambda(\omega) = -\sum_{n=1}^{\infty} c(n) \sin n\omega \tag{9}$$

where $\{c(n)\}$ are the cepstral coefficients. A detailed discussion on the cepstrum and its properties can be found in [6]. Taking the derivative of (9) with respect to ω, we get

$$\theta'(\omega) = -\sum_{n=1}^{\infty} nc(n) \cos n\omega. \tag{10}$$

From (8) and (9) we note that for a minimum phase signal, the spectral magnitude and phase are related through the cepstral coefficients. Further, the group delay function $\tau(\omega)$ $(= -\theta'(\omega))$ can be obtained directly from the cepstral coefficients using (10). For a maximum phase signal (8) is still valid, but (9) and (10) are modified as follows:

$$\theta(\omega) = \theta_v(\omega) + 2\pi\lambda(\omega) = \sum_{n=1}^{\infty} c(n) \sin n\omega \tag{11}$$

$$\theta'(\omega) = \sum_{n=1}^{\infty} nc(n) \cos n\omega. \tag{12}$$

For mixed phase signals we do not have relations of the form (7)-(10) involving a single set of cepstral coefficients. We define two sets of cepstral coefficients $\{c_1(n)\}$ and $\{c_2(n)\}$ for magnitude and phase functions separately as follows:

$$\ln |X(\omega)| = c_1(0)/2 + \sum_{n=1}^{\infty} c_1(n) \cos n\omega \tag{13}$$

and

$$\theta_x(\omega) + 2\pi\lambda(\omega) = -\sum_{n=1}^{\infty} c_2(n) \sin n\omega \tag{14}$$

where $\{c_1(n)\}$ and $\{c_2(n)\}$ are the cepstral coefficients of the minimum phase equivalent signals derived from the spectral magnitude and phase, respectively.

D. Group Delay Functions

Using (10) we can define

$$\tau_m(\omega) = \sum_{n=1}^{\infty} nc_1(n) \cos n\omega \tag{15}$$

as the group delay function derived from the magnitude $|X(\omega)|$ and

$$\tau_p(\omega) = \sum_{n=1}^{\infty} nc_2(n) \cos n\omega \tag{16}$$

as the group delay function derived from the phase $\theta_x(\omega)$. It may be noted that $\tau_m(\omega)$ and $\tau_p(\omega)$ are same as the group delay functions (in the sense usually defined in literature) of the two minimum phase equivalent signals, one derived from the magnitude and the other from phase, respectively. Moreover, $\tau_p(\omega)$ is the usual group delay function of the original signal. For a minimum phase signal $\{x(n)\}$, $\tau_p(\omega) = \tau_m(\omega)$, and for a maximum phase signal, $\tau_p(\omega) = -\tau_m(\omega)$.

The methods to compute $\tau_m(k)$ and $\tau_p(k)$ using an N-point DFT are summarized below [7].

Computation of $\tau_m(k)$:

i) Let $|X(k)|$, $k = 0, 1, \cdots N-1$ be the given spectral magnitude samples.

ii) Obtain $\{c(n)\}$ through the inverse DFT of $\{\ln |X(k)|\}$.

iii) Form the sequence $\{g(n)\}$, where

$$g(0) = 0$$
$$g(n) = nc(n), \qquad n = 1, 2, \cdots, N/2$$
$$g(n) = g(N-n), \qquad n = N/2+1, N/2+2, \cdots, N-1.$$

iv) Obtain $\{\tau_m(k)\}$ through DFT of $\{g(n)\}$.

Computation of $\tau_p(k)$:

i) Obtain an all-pass sequence $\{g(n)\}$ from the given phase samples $\theta(k)$, $k = 0, 1, \cdots, N-1$.

ii) Form the sequence $\{h(n)\}$, where

$$h(0) = 0$$
$$h(n) = ng(n), \qquad n = 0, 1, \cdots, N/2$$
$$h(n) = (n-N)\, g(n), \qquad n = N/2+1, N/2+2, \cdots, N-1.$$

iii) Let the DFT's of $\{g(n)\}$ and $\{h(n)\}$ be $\{G(k)\}$ and $\{H(k)\}$, respectively.

$$G(k) = G_1(k) + jG_2(k), \qquad k = 0, 1, \cdots, N-1.$$
$$H(k) = H_1(k) + jH_2(k), \qquad k = 0, 1, \cdots, N-1.$$

Then

$$\tau_p(k) = G_1(k) H_1(k) + G_2(k) H_2(k), \qquad k = 0, 1, \cdots, N-1.$$

E. Characteristics of $\tau_m(\omega)$ and $\tau_p(\omega)$

In this section, we discuss the characteristics of $\tau_m(\omega)$ and $\tau_p(\omega)$ and their interrelationships for different types of signals. The mean value of $\tau_m(\omega)$ corresponds to the logarithm of the constant scale factor in the magnitude function and the mean value of $\tau_p(\omega)$ corresponds to the slope of the linear phase component in the phase function. In the following discussion we assume that both $\tau_m(\omega)$ and $\tau_p(\omega)$ have zero mean values.

1) Minimum Phase Signals: In Fig. 1 the group delay functions $\tau_m(\omega)$ and $\tau_p(\omega)$ for three different cases of minimum phase signals are shown. For these cases $\tau_m(\omega) = \tau_p(\omega)$.

2) Maximum Phase Signals: For a maximum phase signal, the magnitude and phase functions are related in such a way that $\tau_m(\omega) = -\tau_p(\omega)$. Also, $\tau_m(\omega)$ for a maximum phase signal is the same as that for the corresponding minimum phase signal. From a maximum phase signal, the corresponding minimum phase signal can be derived by simply shifting the poles and zeros outside the unit circle to their conjugate reciprocal locations inside the circle. Hence, $\tau_m(\omega)$ for a maximum phase signal exhibits the same behavior as shown in Fig. 1.

3) Mixed Phase Signals: A mixed phase signal can be thought of as a convolution (in the time domain) of its minimum and maximum phase components. Since convolution in the time domain is equivalent to addition in the group delay domain, the group delay functions of a mixed phase signal can be visualized as superposition of the group delay functions of its min-

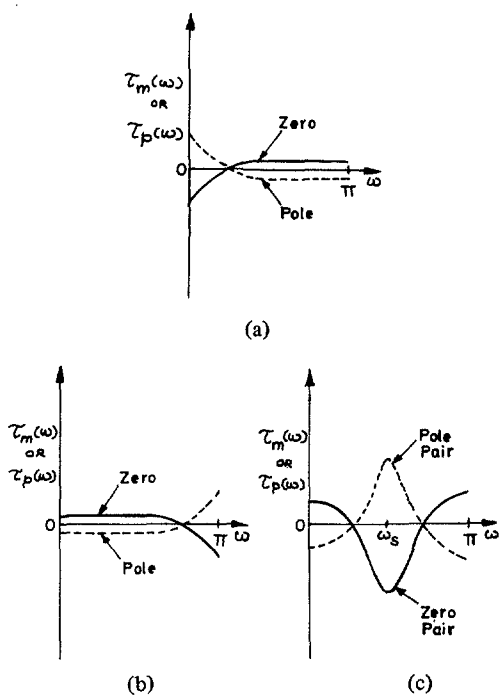


Fig. 1. Group delay functions $\tau_m(\omega)$ and $\tau_p(\omega)$ for minimum phase signals. (a) Signal has a real zero (or pole) at $\omega = 0$. (b) Signal has a real zero (or pole) at $\omega = \pi$. (c) Signal has a complex zero pair (or pole pair) at $\omega = \omega_s$.

imum phase and maximum phase components. The characteristics of $\tau_m(\omega)$ and $\tau_p(\omega)$ for various cases are shown in Fig. 2.

III. Conditions for Signal Reconstruction

The problem of signal reconstruction from the spectral magnitude or phase can be examined on the basis of the characteristics of the group delay functions $\tau_m(\omega)$ and $\tau_p(\omega)$ illustrated in Section II. It can be observed from Fig. 2 that, using $\tau_m(\omega)$ alone, we cannot differentiate between a pole (zero) inside the unit circle and a pole (zero) at a conjugate reciprocal location outside the unit circle in the z plane. Likewise, using $\tau_p(\omega)$ alone, we cannot differentiate between a pole (zero) inside the unit circle and a zero (pole) at conjugate reciprocal location outside the unit circle. These ambiguities are resolved when we are given both $\tau_m(\omega)$ and $\tau_p(\omega)$, which is equivalent to saying that the spectral magnitude and phase together specify the signal uniquely.

Since $\tau_m(\omega)$ and $\tau_p(\omega)$ are equivalent representations of the spectral magnitude and phase respectively, the conditions under which a signal can be reconstructed from magnitude (or phase) can be stated in terms of the conditions under which the signal is uniquely specified by $\tau_m(\omega)$ or $\tau_p(\omega)$. In the following discussion we assume that the signals have no poles or zeros on the unit circle in the z plane. Further, we consider only those signals for which the linear phase component has been removed from the phase function. Whenever a signal is reconstructed from phase it is understood that the reconstruction is correct to within a scale factor.

The conditions for signal reconstruction are discussed below with reference to the illustrations given in Table I.

1) Minimum Phase Condition: The condition that all the poles and zeros of a signal are located inside the unit circle enables us to resolve the ambiguities discussed earlier for $\tau_m(\omega)$ and $\tau_p(\omega)$. Further, $\tau_m(\omega)$ specifies $\tau_p(\omega)$ uniquely and vice versa since both are equal. Hence, a minimum phase signal can be reconstructed completely from its spectral magnitude or phase.

2) Maximum Phase Condition: The condition that all the poles and zeros of a signal are located outside the unit circle enables us to specify the signal uniquely using $\tau_m(\omega)$ or $\tau_p(\omega)$. Further, $\tau_m(\omega)$ specifies $\tau_p(\omega)$ uniquely and vice versa, since $\tau_p(\omega) = -\tau_m(\omega)$. Hence, a maximum phase signal can be reconstructed completely from its spectral magnitude or phase.

3) Mixed Phase Signal with Poles Located Inside the Unit Circle and Zeros Located Outside the Unit Circle: Since all the poles of the signal are constrained to be inside the unit circle and all its zeros outside, $\tau_m(\omega)$ specifies the signal uniquely, provided there are no pole zero pairs occurring at conjugate reciprocal locations. This last condition is essential because a complex conjugate pole pair and a complex conjugate zero pair at reciprocal locations together give rise to a $\tau_m(\omega)$ which is zero at all frequencies, as seen in Fig. 2(g). The conditions stated here are the same as those given in [1, Theorem 7]. Signals satisfying these conditions cannot be reconstructed from the phase because $\tau_p(\omega)$ cannot differentiate between a zero pair outside the unit circle and a pole pair inside the unit circle, occurring at reciprocal locations.

4) Mixed Phase Signal with Poles Located Outside the Unit Circle and Zeros Located Inside the Unit Circle: For reasons analogous to those discussed in 3), this signal can be reconstructed from its spectral magnitude provided there are no pole zero pairs occurring at conjugate reciprocal locations [1, Theorem 8].

5) All-Zero Mixed Phase Signals: Since from $\tau_m(\omega)$ we cannot distinguish between a zero pair inside and outside the unit circle, we cannot reconstruct an all-zero mixed phase signal from its spectral magnitude alone. Such an amgibuity does not arise in $\tau_p(\omega)$, and hence we can reconstruct this signal from its phase, provided there are no complex conjugate zeros occurring at reciprocal locations. The latter condition ensures that no cancellation occurs in $\tau_p(\omega)$ as in Fig. 2(e) [1, Theorems 1, 3, 5].

6) All-Pole Mixed Phase Signal: For reasons similar to those discussed in 5), these signals cannot be reconstructed from the spectral magnitude. However, if there are no complex conjugate pole pairs occurring at reciprocal locations, we can reconstruct these signals from the phase [1, Theorems 2, 4, 6].

7) Mixed Phase Signals with All Poles (Zeros) at Conjugate Reciprocal Locations: These signals can be reconstructed from the spectral magnitude alone, making use of the fact that $\tau_p(\omega) = 0$ [Fig. 2(e) and 2(f)].

8) Mixed Phase Signals with Pole Zero Pairs Occurring at Conjugate Reciprocal Locations: These signals can be reconstructed from the phase alone, making use of the fact that $\tau_m(\omega) = 0$ [Fig. 2(g) and 2(h)].

Examples of group delay functions for cases 1) to 8) are shown in Table I.

In all the cases discussed so far, we have considered only sig-

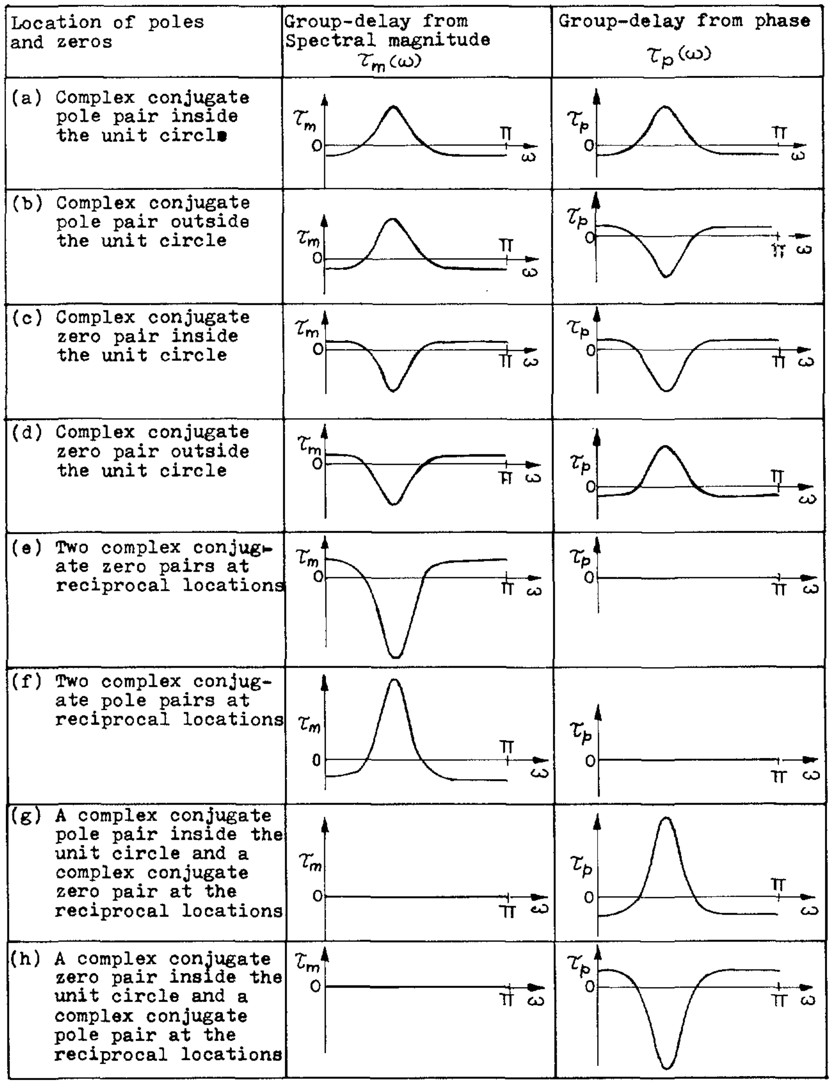


Fig. 2. Group delay functions $\tau_m(\omega)$ and $\tau_p(\omega)$ for different types of signals.

nals whose poles or zeros occur in complex conjugate pairs. However, the arguments are equally valid even for those signals which have zeros (poles) occurring on the real axis. Further, the conditions for signal reconstruction are independent of the multiplicity of poles or zeros at a given location.

IV. Algorithms for Signal Reconstruction

A. Review of Existing Algorithms

1) Iterative Algorithms for Minimum Phase Signals [2]: These algorithms involve repeated transformation between time and frequency domains, with the known constraints imposed in each iteration.

Let $\{x(n)\}$ be the minimum phase signal to be reconstructed from its magnitude (phase) function, and let $\{x_i(n)\}$ be its estimate at the ith iteration. In the frequency domain the magnitude (phase) function of $\{x_i(n)\}$ is replaced by the given magnitude (phase) function. The causality condition is imposed in the time domain to obtain the $(i + 1)$th estimate. The causality condition implies setting the points outside the interval $0 \leq n \leq N/2$ to zero, where, N is the number of points used for DFT. Figs. 3 and 4 illustrate the algorithms for signal reconstruction from magnitude and phase functions, respectively. The signal used at the start of the iterations is usually a zero phase signal in Fig. 3 and an all-pass signal in Fig. 4.

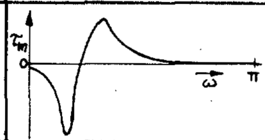
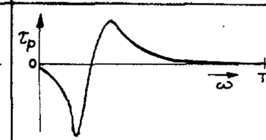
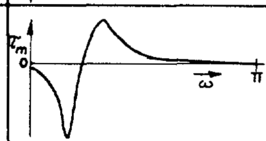
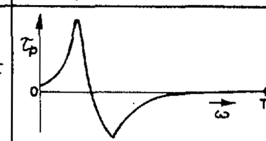
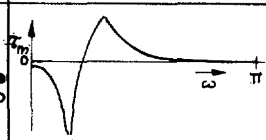
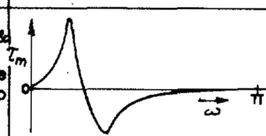
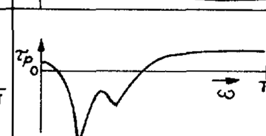
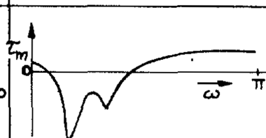
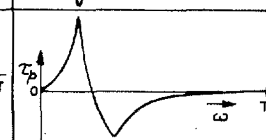
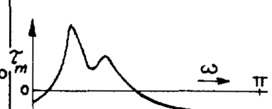
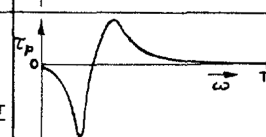
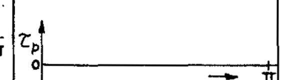
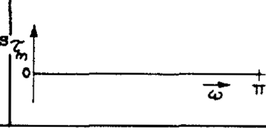
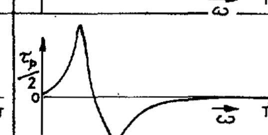
For nonminimum phase signals, these iterative algorithms build up the respective minimum phase equivalent signals.

2) Iterative Algorithm for Reconstruction of a Finite Duration Mixed Phase Signal from Phase Function [1]: This algorithm is identical to the one described in Fig. 4, except that instead of imposing the causality constraint, the known finite duration constraint is imposed. Thus, in the time domain, if it is known that the original signal is confined to the interval $0 \leq n \leq M - 1$, the points outside this interval are set to zero as illustrated in Fig. 5. Here the finite duration condition implies that the signal can be represented using an all-zero model. Thus, it comes under the category of signals described in 5) in Section III. It is interesting to note that if we employ the causality constraint as shown in Fig. 4 instead of the known finite duration, we get the minimum phase equivalent signal derived from the phase of the original signal.

3) Noniterative Algorithm for Minimum Phase Signal Reconstruction [7]: These algorithms, described in [7], are

TABLE I

TYPES OF SIGNALS RECONSTRUCTABLE WITH PARTIAL INFORMATION WITH ILLUSTRATIVE EXAMPLES

Additional information	Group-delay from spectral magnitude $\tau_m(\omega)$	Group-delay from phase $\tau_p(\omega)$	Reconstructable from Magnitude	Reconstructable from Phase
A. Minimum phase			yes	yes
B. Maximum phase			yes	yes
C. Mixed phase: poles inside & zeros outside the unit circle with no reciprocal pole-zero pairs			yes	no
D. Mixed phase: poles outside & zeros inside the unit circle with no reciprocal pole-zero pairs			yes	no
E. Mixed phase: zeros inside & outside the unit circle with no reciprocal zero pairs			no	yes
F. Mixed phase: poles inside & outside the unit circle with no reciprocal pole pairs			no	yes
G. Mixed phase: reciprocal zero pairs & reciprocal pole pairs			yes	no
H. Mixed phase: reciprocal pole-zero pairs			no	yes

Case A: Poles at 0.8 exp $(\pm j60°)$; zeros at 0.866 exp $(\pm j30°)$.
Case B: Poles at 1.25 exp $(\pm j60°)$; zeros at 1.155 exp $(\pm j30°)$.
Case C: Poles at 0.8 exp $(\pm j60°)$; zeros at 1.155 exp $(\pm j30°)$.
Case D: Poles at 1.155 exp $(\pm j30°)$; zeros at 0.8 exp $(\pm j60°)$.
Case E: Zeros at 0.8 exp $(\pm j60°)$ and 1.155 exp $(\pm j30°)$.
Case F: Poles at 0.8 exp $(\pm j60°)$ and 1.155 exp $(\pm j30°)$.
Case G: Poles at 0.866 exp $(\pm j30°)$ and 1.155 exp $(\pm j30°)$ and zeros at 0.8 exp $(\pm j60°)$ and 1.25 exp $(\pm j60°)$.
Case H: Poles at 0.866 exp $(\pm j30°)$ and 1.25 exp $(\pm j60°)$ and zeros at 1.155 exp $(\pm j30°)$ and 0.8 exp $(\pm j60°)$.

shown in Figs. 6 and 7. The reconstruction from magnitude (Fig. 6) involves computation of the cepstral coefficients first, and then the phase function from the odd symmetric sequence of the cepstral coefficients. From the given magnitude and reconstructed phase, the original minimum phase signal can be obtained through the inverse DFT. The reconstruction from the phase (Fig. 7) involves the computation of cepstral coefficients through $\tau_p(k)$ first, and then the magnitude function from the even symmetric sequence of the cepstral coefficients.

It is interesting to note that the first part of the algorithm, namely computation of cepstral coefficients, suggests a method of obtaining unwrapped phase from the given principal value of the phase function [7]. The unwrapped phase is obtained from the odd symmetric sequence of the cepstral coefficients

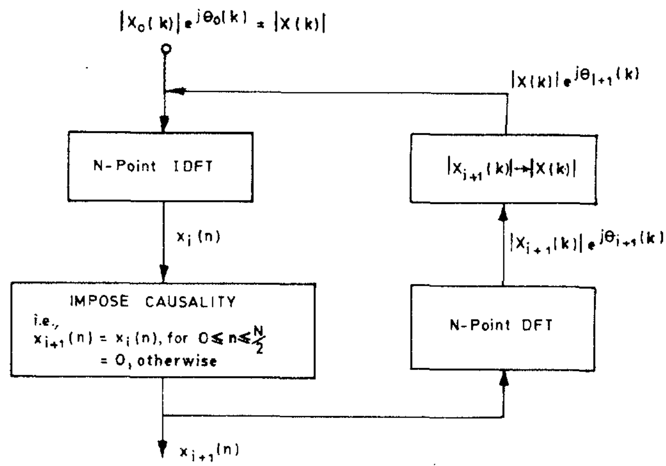


Fig. 3. Iterative algorithm to reconstruct minimum phase signal from its spectral magnitude function.

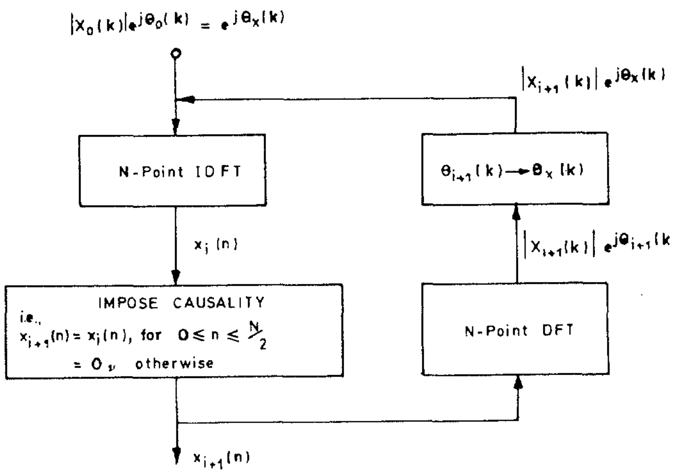


Fig. 4. Iterative algorithm to reconstruct minimum phase signal from its phase function.

through DFT. This phase unwrapping algorithm does not recover the linear and constant phase components. The algorithm is applicable for mixed phase signals also.

From the discussion of the existing algorithms for signal reconstruction it is clear that noniterative techniques given in [7] can be effectively used for minimum phase signal reconstruction, and one need not always resort to the iterative techniques given in [2]. Iterative techniques are definitely useful for the reconstruction of finite duration mixed phase signals from phase.

B. Reconstruction Algorithms for Different Types of Signals Listed in Table I

Here we develop the algorithms for signal reconstruction for the different types of signals given in Table I in terms of known iterative and noniterative techniques. In all these cases we assume that the number of DFT points (N) is sufficiently large, so that aliasing effects are negligible. In our illustrative examples we have used $N = 256$.

1) Minimum Phase Signal: The noniterative algorithms given in Figs. 6 and 7 can be applied for reconstruction of minimum phase signals from spectral magnitude and phase, respectively. For the example given for case A in Table I, reconstructed signals are the same as the original.

2) Maximum Phase Signal: The noniterative algorithms given in Figs. 6 and 7 can be used for maximum phase signals with the modification that the sign of the phase $\theta_x(k)$ in Fig. 6 and the sign of $\tau_p(k)$ in Fig. 7 should be reversed to obtain the correct group delay relations. Since in practice signals do not have poles outside the unit circle, we consider an all-zero maximum phase sequence (-1.0, 1.1, 3.79, -5.5, 42.05, -39.6, 43.56) for illustration. In this case also, the signal reconstructed from magnitude or phase is the same as the original signal.

3) Mixed Phase Signal with Poles Inside and Zeros Outside the Unit Circle: For a mixed phase signal with poles inside and zeros outside the unit circle, the procedure for signal reconstruction from its spectral magnitude function is as follows (Fig. 8):

i) Using the noniterative algorithm of Fig. 6, obtain the phase of the minimum phase equivalent of the original signal from the given spectral magnitude function.

ii) Using the above phase function and a finite duration constraint in the iterative techniques of Fig. 5, obtain an all-zero signal. It may be noted that, for convenience of applying the finite duration constraint, a linear phase function should be added to the phase function obtained in step i) so that the all-zero signal is confined to the interval $0 \leq n \leq M - 1$. The value of time shift n_0 for this linear phase is equal to the number of poles in the original signal. The duration M is equal to the total number of poles and zeros in the original signal, plus one.

iii) Using the magnitude function of this all-zero signal in the noniterative technique of Fig. 6, obtain the phase function $\theta(k)$ of its minimum phase equivalent.

iv) The phase of the original signal is given by $-\theta(k)$. This can be proven as follows.

In the all-zero signal obtained in step ii), the poles of the original signal have been reflected as zeros at conjugate reciprocal locations outside the unit circle. Let $\tau_{m1}(k)$ and $\tau_{m2}(k)$ be the group delay functions obtained from the spectral magnitudes of the original signal and from the all-zero signal obtained in step ii), respectively. Then the maximum phase component of $\tau_{m1}(k)$ (corresponding to zeros of the original signal) is given as

$$[\tau_{m1}(k)]_{\text{max phase}} = (\tau_{m1}(k) + \tau_{m2}(k))/2 \tag{17}$$

and the minimum phase component of $\tau_{m1}(k)$ (corresponding to poles of the original signal) is given as

$$[\tau_{m1}(k)]_{\text{min phase}} = (\tau_{m1}(k) - \tau_{m2}(k))/2. \tag{18}$$

Since the original signal is a convolution of its minimum and maximum phase components, the group delay from phase of the original signal is given by

$$\begin{aligned}\tau_{p1}(k) &= [\tau_{p1}(k)]_{\text{min phase}} + [\tau_{p1}(k)]_{\text{max phase}}\\ &= [\tau_{m1}(k)]_{\text{min phase}} - [\tau_{m1}(k)]_{\text{max phase}}\\ &= \frac{\tau_{m1}(k) - \tau_{m2}(k)}{2} - \frac{\tau_{m1}(k) + \tau_{m2}(k)}{2}\\ &= -\tau_{m2}(k).\end{aligned} \tag{19}$$

This shows that the phase of the original signal and the phase of the minimum phase equivalent of the all-zero signal are equal and opposite.

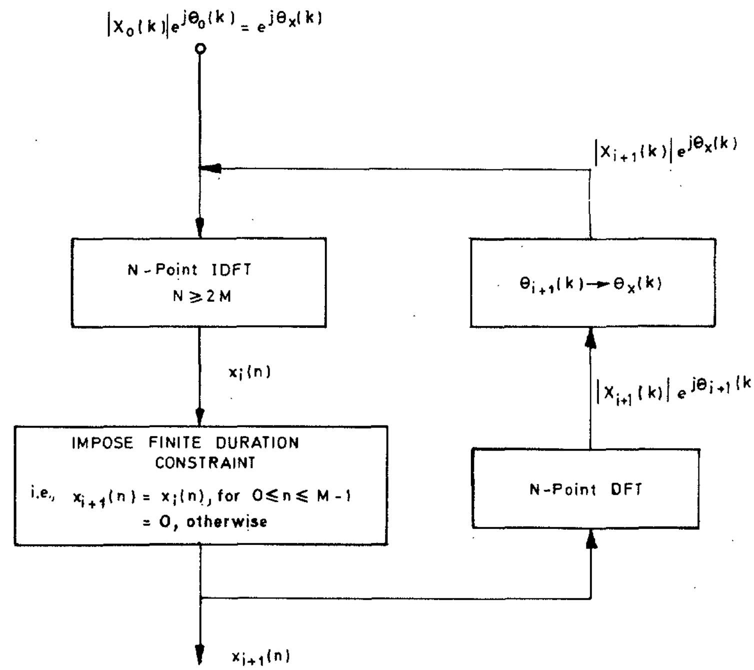


Fig. 5. Iterative algorithm to reconstruct a finite duration mixed phase signal from its phase function.

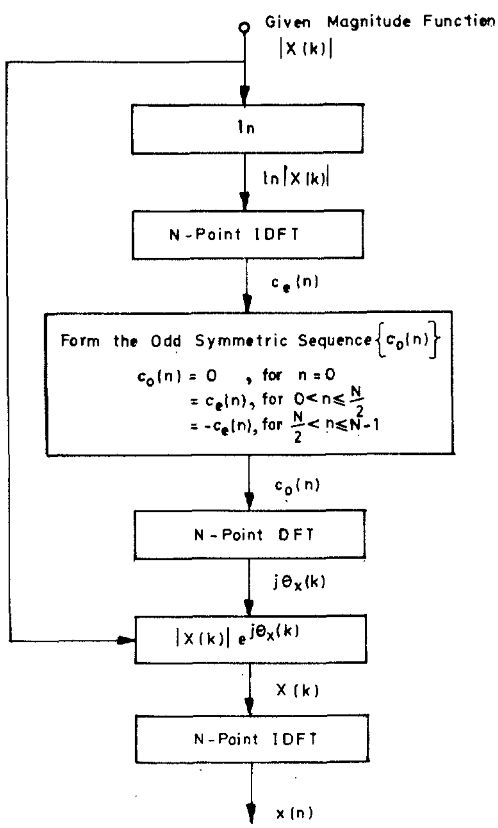


Fig. 6. Noniterative algorithm to reconstruct a minimum phase signal from its spectral magnitude function.

For the example given in case C in Table I, the original signal and the signal reconstructed from its spectral magnitude using the above algorithm were found to be identical. The accuracy of reconstruction through this algorithm depends upon the accuracy of the iterative algorithm of Fig. 5. In our example, 100 iterations were used for the iterative stage of the algorithm.

4) Mixed Phase Signal with Poles Outside and Zeros Inside the Unit Circle: Reconstruction from spectral magnitude in this case can be done in exactly the same manner as in the previous case, except that the phase obtained in step iii) itself gives the phase of the original signal. Thus, the inverse Fourier transform of $|X(k)|\ e^{j\theta(k)}$ in Fig. 8 yields the original signal $\{x(n)\}$. This is only a hypothetical case, since signals with poles outside the unit circle cannot be represented and handled through DFT.

5) Mixed Phase All-Zero Signal: A finite duration mixed phase signal is an example of this category. The signal can be reconstructed from its phase by the iterative technique [1] given in Fig. 5. The all-zero signal for case E in Table I is considered for illustration. The reconstructed signals for different iterations are shown in Table II.

6) Mixed Phase All-Pole Signal: Reconstruction of all-pole mixed phase signal from phase can be done as follows. i) Obtain a new phase function by changing the sign of the given phase function. ii) Use this phase in the iterative algorithm in Fig. 5 to obtain a spectral magnitude function. The finite duration to be used in this case is the duration of the convolutional inverse of the original signal. iii) The reciprocal of the spectral magnitude function gives the spectral magnitude function of the original signal.

7) Mixed Phase Signal with Conjugate Reciprocal Pole (Zero) Pairs: For this type of signal, $\tau_p(\omega) = 0$. Therefore, the original signal is the zero phase signal obtained from the given magnitude function.

8) Mixed Phase Signal with Conjugate Reciprocal Pole-Zero Pairs: For this type of signal $\tau_m(\omega) = 0$. Therefore, the original signal is the all-pass signal obtained from the given phase function.

In the illustrative examples for signal reconstruction we have considered so far, the original signal had poles or zeros occurring only in complex conjugate pairs. However, the algorithms are applicable even when the signal has zeros or poles on the real axis, or if there are multiple poles or zeros. Also, the pres-

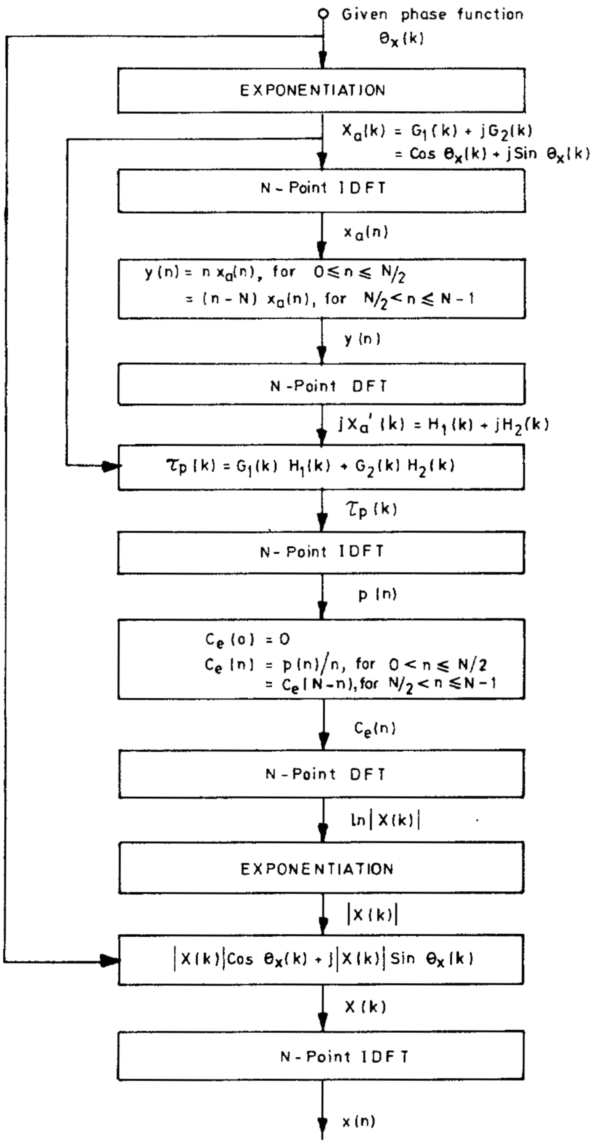


Fig. 7. Noniterative algorithm to reconstruct a minimum phase signal $\{x(n)\}$ from its phase function.

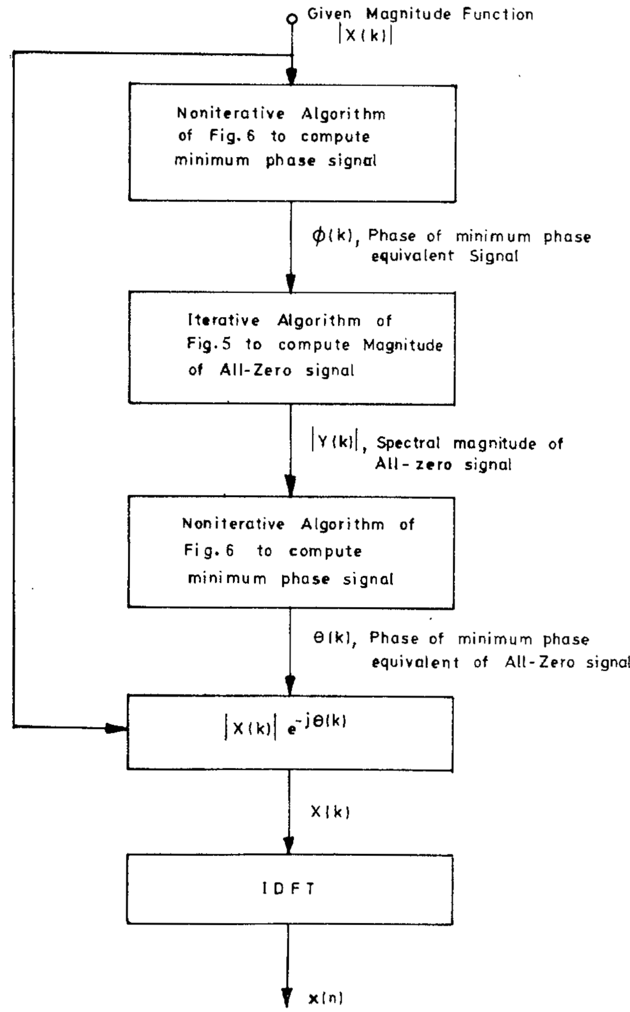


Fig. 8. Algorithm to reconstruct a mixed phase signal (with poles inside and zeros outside the unit circle) from its spectral magnitude.

ence of more than one pole or zero or both at the same frequency does not impose any limitation on these algorithms. Examples of such signals for which the signal reconstruction algorithms were verified are given below.

i) The signal sequence (1, -1.6, 0.55) is a mixed phase all-zero sequence with two zeros on the real axis, one outside the unit circle, and one inside the unit circle, i.e., it has both the zeros at the same frequency. This signal was reconstructed from phase using the iterative technique given in Fig. 5.

ii) The signal sequence (1, -1.6, 1.92, -1.024, 0.4096) which is a minimum phase signal having double zeros at $z = 0.8\ e^{\pm j60^\circ}$ was reconstructed from magnitude as well as from phase using the noniterative algorithms given in Figs. 6 and 7, respectively.

iii) The signal having a pole at $z = 0.5$ and a zero at $z = 0.9$ was reconstructed from magnitude as well as from phase by the noniterative algorithms given in Fig. 6 and 7, respectively. This signal has a pole and zero at the same frequency.

V. Reconstruction of Practical Signals

In the preceding sections we have discussed the theoretical basis for signal reconstruction and illustrated the reconstruction algorithms for some synthetic examples. In this section we consider application of these algorithms for practical signals. In order to reconstruct a signal from magnitude or phase, one should have some knowledge of the nature of the signal so that appropriate algorithms can be applied. We consider speech signals and picture signals for illustration.

Speech is produced as a result of excitation of the vocal tract system by quasiperiodic glottal pulses or by noise excitation or a combination of both. Therefore it is reasonable to assume that speech belongs to the minimum phase category, i.e., case A in Table I and/or to the mixed phase category, i.e., case C in Table I. From a perceptual angle, the spectral magnitude information is more important than the phase information, and hence reconstructing a speech signal using the algorithms corresponding to case A or case C preserves the necessary information for speech intelligibility. Reconstruction from phase alone, however, yields a magnitude function that has peaks at the frequencies of zeros outside the unit circle. Thus, a zero outside the unit circle will be represented as a pole in the mag-

TABLE II
ITERATIVE RECONSTRUCTION OF A MIXED PHASE ALL-ZERO SIGNAL

		x(0)	x(1)	x(2)	x(3)	x(4)
ORIGINAL SIGNAL		1.0	-2.8	3.5733	-2.3467	.8533
Signal reconstructed from phase after	0 iterations	-.8537	-1.7543	3.2307	.3970	1.0018
	10 iterations	.0227	-2.9170	3.8143	-1.6104	.6746
	50 iterations	.9855	-2.8020	3.5733	-2.3416	.8433
	100 iterations	.9998	-2.8000	3.5733	-2.3468	.8533

nitude of the reconstructed signal, which is definitely unacceptable from a perceptual point of view. On the other hand, the only error that occurs in the reconstruction from spectral magnitude of a speech signal is that the phase components corresponding to the maximum phase zeros are changed in sign, which may affect the tonal quality, but not the intelligibility.

Fig. 9 shows cepstrally smoothed log magnitude and group delay functions $\tau_m(\omega)$ and $\tau_p(\omega)$ for two consecutive segments (each 6.4 ms) of voiced speech. It can be seen that the features of the log magnitude function that exist in $\tau_m(\omega)$ are absent in $\tau_p(\omega)$. Specifically, $\tau_m(\omega)$ has peaks and valleys corresponding to peaks and valleys of the log magnitude function, whereas $\tau_p(\omega)$ exhibits peaks at some frequencies where there are valleys in the log magnitude function. Hence, for spectrographic displays, which represent the log magnitude function in successive time frames, it is the magnitude function that is more important.

Picture signals with finite support (the two-dimensional equivalent of finite duration) may be included in the category of all-zero signals, i.e., case E in Table I. We notice for this case that there is ambiguity in the group delay function derived from the magnitude due to presence of zeros outside and inside the unit circle. This ambiguity is absent in the group delay function derived from the phase function. This makes it possible for finite support picture signals to be reconstructed from only the phase function [3], [4]. On the other hand, for reconstruction of such sequences from the magnitude function, we need additional information such as one bit of phase [3], [8] or the boundary conditions of the signal [9] to resolve the ambiguity caused by the simultaneous presence of minimum phase and maximum phase zeros.

An example of picture signal reconstruction from the phase function can be seen in Fig. 10. Fig. 10(a) shows the original picture and Fig. 10(b) is the all-pass signal obtained from phase function. Fig. 10(c) and 10(d) shows the reconstructed images after 10 and 50 iterations, respectively. The constraints used in the spatial domain are i) finite support and ii) limited dynamic range (i.e., the pixel values are constrained to be between a minimum which is generally zero, and a maximum which is generally 255 for an 8 bit/pixel representation). The clarity of the reconstructed image improves significantly with iterations.

Fig. 11(b) shows a blurred image obtained by low-pass filtering the image in Fig. 11(a) with a linear phase filter. The phase only reconstruction of this image is shown in Fig. 11(c). With successive iterations the reconstructed image does not converge to the image in Fig. 11(b). This is due to the fact that changes in the spectral magnitude caused by the low-pass filtering do not affect the phase and the phase only reconstruction ignores the linear phase factor.

Fig. 12(b) shows a noisy image obtained by adding noise to the picture in Fig. 12(a). The phase only reconstruction is shown in Fig. 12(c). With successive iterations the reconstructed image converges to the noisy image in Fig. 12(b). This is because additive noise affects the phase, and hence affects the image reconstructed from phase.

Fig. 13 illustrates an attempt to reconstruct the image from the spectral magnitude function. The image in Fig. 13(b) is constructed from the spectral magnitude of Fig. 13(a) and zero phase. If we use an iterative technique to reconstruct the image from its spectral magnitude, imposing a finite support constraint, we obtain the minimum phase equivalent of the original image, i.e., the maximum phase zeros of the original image get reflected to the reciprocal locations inside the unit circle. This is equivalent to convolving the minimum phase component of the original image with the inverted version of the maximum phase component which yields a confusing image as shown in Fig. 13(c).

VI. CONVERGENCE PROCESS IN ITERATIVE RECONSTRUCTION ALGORITHMS

Group delay functions suggest an interesting explanation for the process of convergence of the iterative algorithms described in Section IV. Consider a signal having a complex conjugate zero pair inside the unit circle at a frequency location A, and a complex conjugate zero pair outside the unit circle at a frequency location B. The group delay functions $\tau_p(\omega)$ and $\tau_m(\omega)$ for this signal are shown in Fig. 14(a). The figure also shows the group delay functions $\tau_{m\,\min}(\omega)$ and $\tau_{m\,\max}(\omega)$ of the minimum and maximum phase components of the signal, where

$$\tau_{m\,\min}(\omega) = [\tau_m(\omega) + \tau_p(\omega)]/2 \tag{20}$$

$$\tau_{m\,\max}(\omega) = [\tau_m(\omega) - \tau_p(\omega)]/2. \tag{21}$$

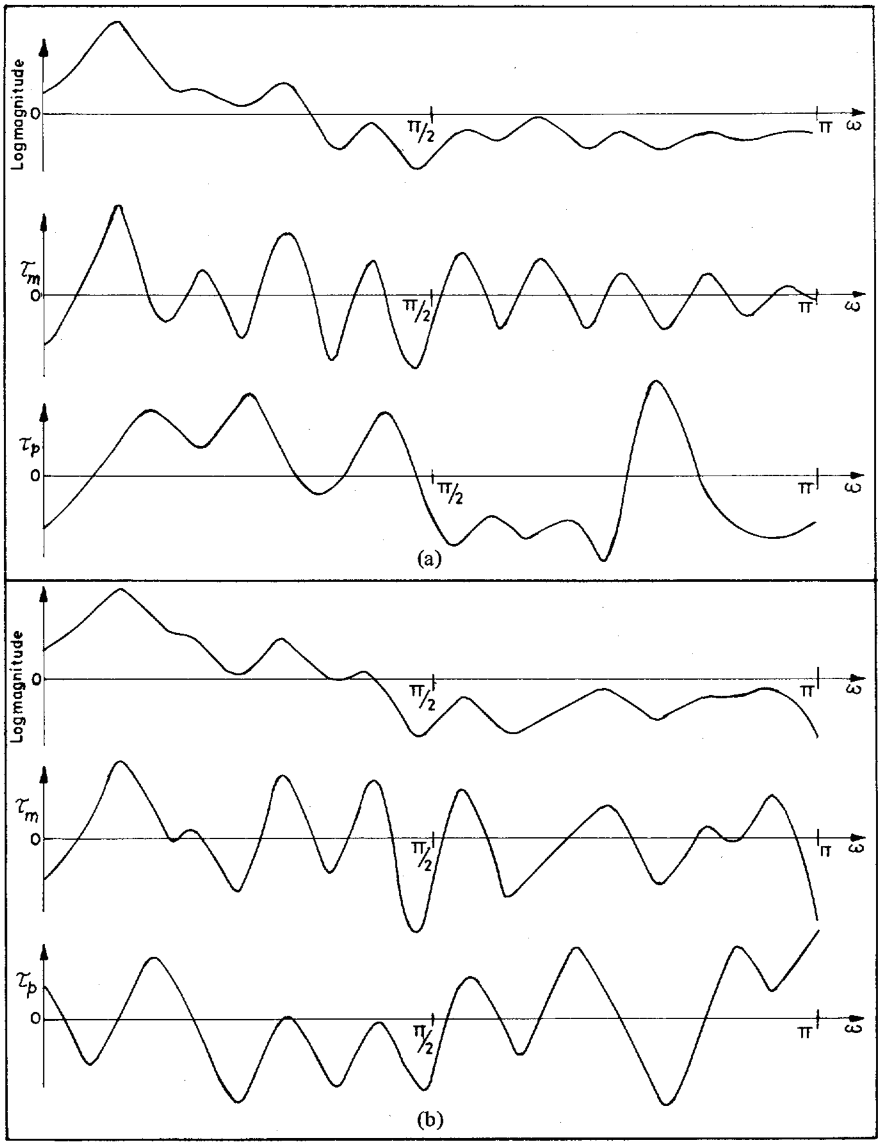
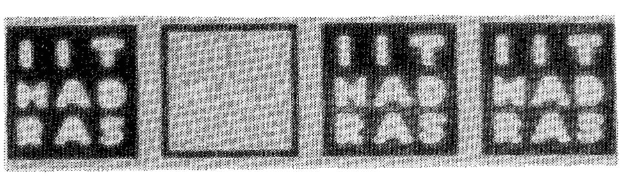
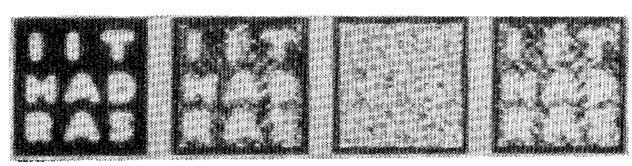


Fig. 9. Cepstrally smoothed log spectrum and group delay functions $\tau_m(\omega)$ and $\tau_p(\omega)$ for two consecutive segments (each 6.4 ms) of voiced speech. (a) Segment 1. (b) Segment 2.



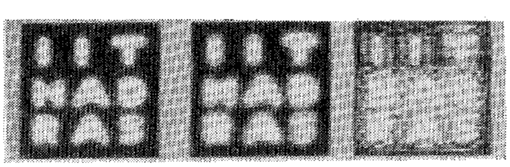
(a) (b) (c) (d)

Fig. 10. Iterative reconstruction of an image signal from phase. (a) Original image. (b) All-pass image. (c) Reconstructed image after 10 iterations. (d) Reconstructed image after 50 iterations.



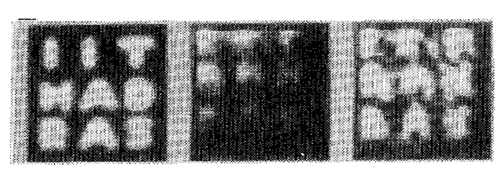
(a) (b) (c) (d)

Fig. 12. Result of iterative phase only reconstruction of image degraded by additive noise. (a) Original image. (b) Degraded image obtained by adding noise to the image in Fig. 12(a). (c) All-pass image constructed with the phase of the image in Fig. 12(b). (d) Phase only reconstruction after 50 iterations.



(a) (b) (c)

Fig. 11. Result of iterative, phase only reconstruction of a blurred image. (a) Original image. (b) Blurred image obtained by low-pass filtering the image in Fig. 11(a). (c) Reconstruction from the phase of the blurred image in Fig. 11(b) after 50 iterations.



(a) (b) (c)

Fig. 13. Reconstruction of an image signal from spectral magnitude function. (a) Original image. (b) Zero phase image with the spectral magnitude of the image in Fig. 13(a). (c) Image obtained after 10 iterations.

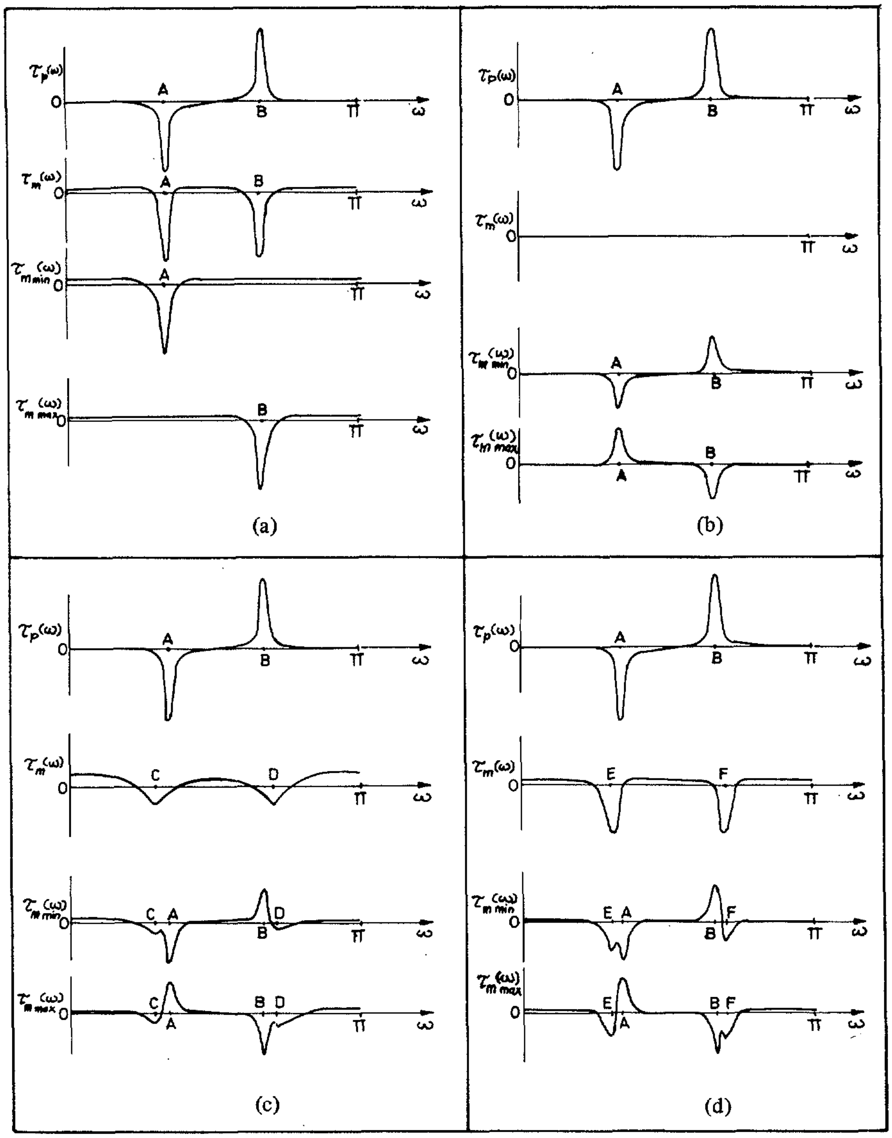


Fig. 14. Group delay functions illustrating the convergence process of iterative reconstruction algorithms. (a) For a mixed phase all-zero signal. (b) For the all-pass signal of Fig. 14(a). (c) For the signal after one iteration. (d) For the signal after two iterations.

As expected $\tau_{m\,\min}(\omega)$ has a negative peak at $\omega = A$ and $\tau_{m\,\max}(\omega)$ has a negative peak at $\omega = B$.

To start with, we obtain an all-pass signal using the phase function of the original signal. The group delay function $\tau_m(\omega)$ of this all-pass signal is zero, and the group delay functions $\tau_{m\,\min}(\omega)$ and $\tau_{m\,\max}(\omega)$ which are obtained using (20) and (21) are equal and opposite, as illustrated in Fig. 14(b). It can be seen that apart from the zeros of the original signal, spurious poles have been created at the reciprocal locations corresponding to zeros at $\omega = A$ and $\omega = B$. Truncation of this all-pass signal to the correct duration gives an all-zero signal which has a $\tau_m(\omega)$ as shown in Fig. 14(c). The frequencies of zeros, i.e., C and D of this signal, are slightly offset from those of the original signal (i.e., A and B).

Combining the spectral magnitude of the truncated all-pass signal with the original phase, we can obtain the signal of the second iteration. The group delay functions $\tau_{m\,\min}(\omega)$ and $\tau_{m\,\max}(\omega)$ of its minimum and maximum phase components are shown in Fig. 14(c). The figure shows that in $\tau_{m\,\min}(\omega)$ there is a zero pair at $\omega = D$ close to the spurious pole pair at $\omega = B$ and, in $\tau_{m\,\max}(\omega)$, there is a zero pair at $\omega = C$ close to the spurious pole pair at $\omega = A$. Thus, spurious poles have been created once again, but this time their effect is partly countered by additional zeros at nearby frequency locations. Truncation of this signal yields an all-zero signal, whose zeros are located at frequencies E and F, which are closer to those of the original signal than C and D. Thus in each iteration, truncation moves the locations of zeros towards the correct values, and simultaneously they try to reduce the effect of the spurious poles more effectively. After a large number of iterations, $\tau_m(\omega)$ reaches the true value given in Fig. 14(a).

Analogously, we can explain the convergence mechanism of the iterative technique for minimum phase signals, where the causality constraint is imposed on the signal at each iteration.

As a consequence, if we remove the linear phase component from the phase of an all-zero mixed phase signal and use it in the iterative algorithm using the causality constraint, we obtain the minimum phase equivalent of the original signal. Conversely, if we add an appropriate linear phase term to the phase of a pole zero minimum phase signal and use it in the iterative technique, employing an appropriate finite duration constraint, we obtain an equivalent finite duration signal, in which the poles of the original signal are reflected as zeros at the conjugate reciprocal locations outside the unit circle.

VII. Summary and Conclusions

In this paper, we have attempted to unify the problems of signal reconstruction from spectral magnitude function alone and from phase function alone through the use of group delay functions. The conditions obtained for signal reconstruction are essentially a restatement of the conditions embodied in the theorems in [1]. However, the interpretation of the theorems in the group delay domain leads to a better conceptual understanding of the duality between spectral magnitude and phase information in signals, and provides the basis for some reconstruction algorithms.

We have reviewed the existing algorithms for reconstruction from spectral magnitude or phase of a minimum phase signal and the algorithm for reconstruction from phase of a mixed phase all-zero signal. We have observed that one need not resort to the iterative techniques in the case of minimum phase signals, since noniterative techniques can be used with advantage. Using a combination of iterative and noniterative techniques, we have developed an algorithm for reconstruction of a mixed phase signal from its spectral magnitude. These ideas can be applied for pole zero decomposition or minimum-maximum phase decomposition of a signal in certain cases.

Based on the conditions for signal reconstruction, we have observed that the relative importance of spectral magnitude and phase information in signals depends on the nature of the signal. We have demonstrated that phase information is more important for a picture signal, whereas for speech signals the short term spectral magnitude information is more important. These results show that one cannot say that phase is more important than spectral magnitude in all cases [4].

To justify the importance of phase, it was argued [4] that the number of bits required to code the phase would be more than that required to code the spectral magnitude for a given mean-square error. This argument may not be valid for a general signal due to the duality we noticed between magnitude and phase information, as seen in the group delay domain. The shapes of the functions $\tau_m(\omega)$ and $\tau_p(\omega)$ suggest that both of the functions may require the same number of bits for coding.

Through group delay functions we have been able to provide an explanation for the convergence process in the iterative algorithms for signal reconstruction.

We have discussed algorithms for reconstruction from spectral magnitude or phase for signals which satisfy the conditions stipulated for reconstruction. The additional information required to handle practical data depends on our knowledge of the signal being reconstructed. Moreover, such knowledge is mainly signal dependent and hence cannot be generalized for all problems. One effort in this direction is the reconstruction of all-zero mixed phase signals from spectral magnitude with the additional information of one bit phase [3], [8] or the boundary values [9]. This additional information for images resolves the ambiguity in the magnitude function between the minimum phase and maximum phase zeros under certain conditions.

Other interesting topics for further investigation are pole-zero decomposition [10] and minimum-maximum phase component decomposition suggested by the group delay functions. This may be eventually lead to very effective algorithms for reconstruction of signals with partial information.

References

[1] M. H. Hayes, J. S. Lim, and A. V. Oppenheim, "Signal reconstruction from phase or magnitude," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-28, pp. 672-680, Dec. 1980.

[2] J. E. Quatieri and A. V. Oppenheim, "Iterative techniques for minimum phase signal reconstruction from phase or magnitude," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-29, pp. 1187-1193, Dec. 1981.

[3] M. H. Hayes, "The reconstruction of a multidimensional sequence from phase or magnitude of its Fourier transform," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-30, pp. 140-154, Apr. 1982.

[4] A. V. Oppenheim and J. L. Lim, "Importance of phase in signals," *Proc. IEEE*, vol. 69, pp. 529-541, May 1981.

[5] B. Yegnanarayana, "Design of recursive group-delay filters by autoregressive modeling," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-30, pp. 632-637, Aug. 1982.

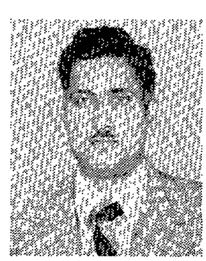
[6] D. G. Childers, D. P. Skinner, and R. C. Kemerait, "The cepstrum: A guide to processing," *Proc. IEEE*, vol. 65, pp. 1428-1443, Oct. 1977.

[7] B. Yegnanarayana and A. Dhayalan, "Noniterative techniques for minimum phase signal reconstruction from phase or magnitude," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, Boston, MA, 1983, pp. 639-642.

[8] P. L. Vanhove, J. S. Lim, and A. V. Oppenheim, "Signal reconstruction from Fourier transform amplitude," in *Proc. SPIE 26th Ann. Int. Tech. Symp.*, San Diego, CA, Aug. 22-27, 1982.

[9] M. H. Hayes and T. F. Quatieri, "The importance of boundary conditions in the phase retrieval problem," in *Proc. ICASSP-82*, pp. 1545-1548.

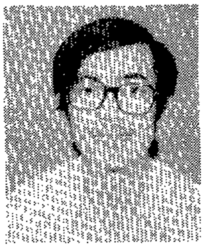
[10] B. Yegnanarayana, "Pole-zero decomposition of speech spectra," Carnegie-Mellon Univ., Pittsburgh, PA, Rep. CMU-CS-79-101, Jan. 1979.



B. Yegnanarayana was born in India on January 9, 1944. He received the B.E., M.E., and Ph.D. degrees in electrical communication engineering from the Indian Institute of Science, Bangalore, India, in 1964, 1966, and 1974, respectively.

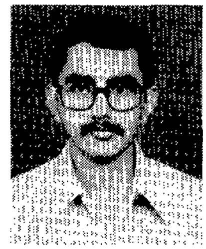
He was a Lecturer from 1966 to 1974 and an Assistant Professor from 1974 to 1978 in the Department of Electrical Communication Engineering, Indian Institute of Science. From 1977 to 1980 he was a Visiting Associate Professor of Computer Science at Carnegie-Mellon University, Pittsburgh, PA. He was a Visiting Scientist at ISRO Satellite Centre, Bangalore, from July to December 1980. He is presently a Professor in the Department of Computer Science and Engineering, Indian Institute of Technology, Madras. During the period from 1966 to 1971 he was engaged in the development of environmental test facilities for the Acoustics Laboratory at the Indian Institute of Science. Since 1972 he has been working on problems in the area of speech signal processing. He is presently engaged in research activities in digital signal processing and speech recognition.

Dr. Yegnanarayana is a member of the Computer Society of India.



D. K. Saikia was born in India on January 7, 1959. He received the B.E. degree in electronics and communication engineering from Madras University, Madras, India, and the M. Tech. degree in electrical engineering from Indian Institute of Technology, Madras, in 1981 and 1983, respectively.

At present he is working as a Research and Development Engineer with Hindustan Computers Ltd., New Delhi, India. His interests are in the fields of digitial signal processing and digital computer hardware.



T. R. Krishnan was born in India on September 2, 1960. He received the B.Tech degree in electrical engineering from the Indian Institute of Technology, Madras, in 1983.

He is at present a graduate student in the Department of Computer Science, University of Wisconsin, Madison.

High Quality Synthesis of Musical Voices in Discrete Time

THOMAS W. GOEDDEL, MEMBER, IEEE, AND STEVEN C. BASS, MEMBER, IEEE

Abstract—**A new technique for high quality syntheses of musical voices (instruments) is presented. It is a harmonic synthesis method based on weighted window function "pulse" sequences. The technique eliminates the problem of aliasing in its discrete-time implementation. High fidelity emulations of "natural" instruments are achieved at a greatly reduced cost in multiplies per sample time. In addition, parameter update rates can be as much as an order of magnitude lower than those required by competing methods.**

I. Introduction

ADVANCES in solid state circuits have caused increasing attention to be given to the application of digital techniques to many signal processing tasks. Among these is the generation of musical sounds. Of particular interest in this paper are the real-time applications that could be made in electronic organs. These machines are sold for home, theatre, church, or professional use, and occupy the \$500 to \$40 000 price range. Analog processing methods still dominate this industry, and subtractive synthesis is the algorithm most commonly used to render the wide range of complex voices which these machines are capable of sounding. Subtractive synthesis within these organs begins with the production of several harmonically-rich excitation signals. To simplify the implementation, these excitation signals are typically rectangular pulse trains. The sole difference between these signals is their respective frequencies: each signal renders a distinct octave of the desired fundamental frequency. Organ engineers commonly refer to these signals as various "footages" of the same basic frequency. For example, four rectangular pulse trains may be produced: the 16′, 8′, 4′, and 2′ versions are most common. If heard separately, these would sound at one, two, four and eight times the desired fundamental frequency, respectively. These separate footage versions would then be "mixed-down" to a single channel, with each footage being independently weighted prior to contributing to the sum. Overall, this procedure applies a coarse harmonic weighting that is appropriate to the instrument or sound being emulated. After this, all simultaneously sounding frequencies (notes) are summed and passed to a fixed formant (voicing) filter. Here, emphasis is applied to the sound components solely on the basis of their frequencies rather than their harmonic numbers.

When considering the implementation of electronic organs in digital hardware, a first thought is to transcribe subtractive syntheses into the digital world, and this is how our work in this area originally began. However, as our initial thoughts were developed and extended to allow increasingly accurate simulations of natural instruments, it became clear that our approach had evolved into a new form of "waveform synthesis." This new method is powerful enough to yield *very* convincing emulations of existing sounds as well as to render

Manuscript received October 26, 1982; revised January 16, 1984. A portion of this work was supported by Kimball Piano and Organ Co., Jasper, IN.

T. W. Goeddel was with the Department of Electrical Engineering, Purdue University, West Lafayette, IN. He is now with AT&T Information Systems, Inc., Neptune, NJ.

S. C. Bass is with the Department of Electrical Engineering, Purdue University, West Lafayette, IN 47907.

0096-3518/84/0600-0623\$01.00 © 1984 IEEE