# ALGORITHMS FOR IMAGE RECONSTRUCTION FROM QUANTIZED PHASE

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by

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## CERTIFICATE

This is to certify that the Thesis entitled **"ALGORITHMS FOR IMAGE RECONSTRUCTION FROM QUANTIZED PHASE"** is the bonafide work of Mr. Pramod Saini, carried out under my guidance and supervision, at the Department of Computer Science and Engineering, Indian Institute of Technology, Madras, for the award of the degree of **MASTER OF SCIENCE** in Computer Science.

(B.Yegnanarayana)

To

**All the good people**

**Of the past, present, and future.**

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**ALGORITHMS FOR IMAGE RECONSTRUCTION FROM QUANTIZED PHASE**

**CONTENTS**

## ABSTRACT

In many practical situations, it is required to obtain information from an array of complex data. Some examples of these situations are - computer aided tomography, medical imaging, acoustic imaging, and synthetic aperture radar (SAR) techniques. In all these cases, data collected by an array of receiver elements is used to derive the required information. The outputs vary according to the situation. The required information may be an image as in case of acoustic imaging or a set of parameters characterizing the source as in SAR applications. Recovery of the required information may involve a straightforward transformation or a more complex procedure.

There are certain practical issues common to problems of this nature. Some issues of interest are - (1) to overcome the complexity of measurements at the receiver array, (2) to recover the information from a limited number of data samples, and (3) to overcome the effects of noise in the data.

In this thesis, we address these issues for a certain class of information recovery problems. We propose algorithms for information recovery from available data and develop theoretical justification for these algorithms.

We consider the reconstruction problem as a problem of information recovery from partial data. We show that it is possible to combine partial information in various domains to recover the required information. The received data is

usually a set of complex numbers. We develop algorithms for signal reconstruction from only the full/quantized phase information of the received data. This reduces the measurement complexity at the receiver end. The phase information represents partial information at the receiver end. It is used alongwith some apriori information (such as finite support constraint) in the signal domain, in an iterative algorithm for signal reconstruction. It was shown that by measuring data at several frequencies, the receiver size can be reduced in terms of the number of receiver elements. We show in this thesis that by using phase quantization with the multiple frequency data, both the measurement and size complexities of the receiver can be reduced, but at the cost of significantly increased computation. Reconstruction from the quantized phase information also reduces the effects of noise in the measured data.

The algorithms proposed in this thesis can be applied to a wide range of information recovery problems. In this work, we address the problem with special reference to acoustic imaging.

Acoustic imaging is a technique of image formation using acoustic waves. We consider the holographic acoustic imaging method in our studies because it allows the use of signal processing techniques at the data processing stage.

Like other problems of this category, the main issues in acoustic imaging are to find techniques to reduce the circuit and measurement complexity of the setup and to improve the image quality. We study the use of the algorithms proposed in this thesis to achieve these goals. The experimental results show that quantized phase measurements at several frequencies help to trade the measurement and size complexity with the computational complexity.

# CHAPTER 1

# PARTIAL DATA PROBLEMS

## 1.1 INTRODUCTION TO INFORMATION RECOVERY PROBLEMS

In many practical situations, it is required to obtain information from an array of complex data. Some examples of these situations are - computer aided tomography, medical imaging, acoustic imaging, and synthetic aperture radar (SAR) techniques. In all these cases, the data collected by an array of receiver elements is used to derive the required information. The outputs vary according to the situation. The required information may be an image as in the case of acoustic imaging or a set of parameters characterizing the source as in SAR applications.

The data measured in these problems is usually a transformation of the original signal. Therefore signal reconstruction is done by computing the inverse transform on the received data. As we shall see later, the process of inverse transform may be more complex than computing just a Fourier transform.

There are certain practical issues common to problems of this nature. Some issues of interest are - (1) to overcome the complexity of measurements at the receiver array, (2) to recover the information from a limited number of data samples, and (3) to overcome the effects of noise in the data. The data is measured by an array of receiver elements. The number of elements on the receiver array is finite.

Generally the data is a set of complex values and has both phase and magnitude components. But due to practical problems, it may not be possible to make measurement of both the phase and magnitude accurately. Moreover, the signal at the receiver end is usually noisy. In all these cases the information may be considered incomplete because of the finite and discrete measurements, or because of some missing phase/magnitude values. Noise causes ambiguity in each data value. Therefore we can consider the available data as partial information. The aim of this work is to propose techniques to solve these partial data problems.

The partial data problem is encountered in many real life situations also. It is such an integral part of nature that all human beings make inferences from partially available information in nearly all aspects of life. This includes common activities like listening, seeing, reading, etc. Recovery from partial data is possible because usually the domain of interest has many redundancies. Some apriori information can be used to recover the complete information from partial data. Human beings use common intelligence, past experience, and the accumulated knowledge in such situations. For example, while listening to unclear speech, such as on telephone, we make use of the recognizable words, the context, etc., along with the sound and duration of the unclear words to recognize them and understand the whole sentence. In a similar way we make use of the context and apriori knowledge to recognize objects when they are only partly visible. Misprints or **misspelt** words are easily overlooked while reading fast because we usually read by looking at the overall shape and meaning of the words and word sequences and not their spellings. For example, in the following sentence

**There is a spelling mistake in this sentense.**

one usually overlooks the fact that the last word should have been **'sentence'** and not **'sentense'.** In the same way if a word of the text is partially rubbed off, or is not visible due to some other reasons, it can be guessed most of the times.

Just as human beings solve the partial data problems in daily life, we want machines to do so for some of the cases mentioned in the beginning. In this work we propose techniques to overcome some of the practical constraints for such a class of partial data problems. We show that, with suitable algorithms, it is possible to combine partial information in various domains to recover the required information. When the available data is partial, the missing parts can generally be filled up in a variety of ways. This defines a set of possible solutions. Most of the methods available to solve these problems use information in various domains to limit this set to a small size, and finally pick out the most probable solution from it.

We develop algorithms for signal reconstruction from only the **full/quantized** phase information of the received data. This technique takes care of the situation when only the phase data is available at the receiver end. The measurement complexity at the receiver end can also be reduced with this technique. Only the full/quantized phase information represents partial information at the receiver end. It is used in an iterative algorithm, alongwith some apriori information in the signal domain (such as finite support constraint), for signal reconstruction. It was shown that by measuring data at several frequencies, the receiver size can be reduced in terms of the number of receiver elements. In this thesis, we show that by using phase quantization with the multiple frequency data, both the measurement and size complexities of the receiver can be reduced, but at the cost of significantly increased computation. Reconstruction from the quantized phase information also reduces the effects of noise in the measured data.

Algorithms have been developed in this thesis with special reference to simulated acoustic imaging systems. The aim in acoustic imaging is to form images from the acoustic field data collected by an array of hydrophones. In case of image signals, edge information is very important for understanding the picture or the scene. This is so because objects are recognized from their features. In most of the cases, edges are enough to convey the information about the features. Extensive work has been done to study the relative importance of the Fourier transform phase and the Fourier transform magnitude for reconstruction of the image signals. It has been observed that the Fourier transform phase preserves most of the edge information of a picture. In fact

Fourier transform phase information is sufficient to recover the original signal in many cases. Therefore, images reconstructed from only their Fourier transform phase information have been found to be better than those reconstructed from only their Fourier transform magnitude information. We show that in the class of problems addressed in this thesis, reconstruction is possible from full/quantized phase information. The phase is important for recovery of edge information, and hence, for the recovery of the essential features for recognition of the object. We make use of this property to propose techniques for reducing the circuit and measurement complexity in an acoustic imaging setup.

## 1.2 ACOUSTIC IMAGING : AN EXAMPLE OF PARTIAL DATA PROBLEM

Acoustic imaging is the technique of mapping objects with acoustic radiation [1],[2]. Fig.1.1 shows a typical acoustic imaging set up. Acoustic waves are transmitted from one end. They hit the object which is to be imaged, and the reflected acoustic field is measured at the receiver end. This data is processed to form the image of the original object. Acoustic imaging finds application in underwater imaging, medical imaging, etc., where other sources of radiation, like light, cannot penetrate to the required distances. There are three main approaches to acoustic imaging, namely (a) focussed acoustic imaging, (b) beamforming, and (c) acoustic holography [2]. We restrict our attention to the acoustic holographic approach, since it enables us to use sophisticated signal processing techniques

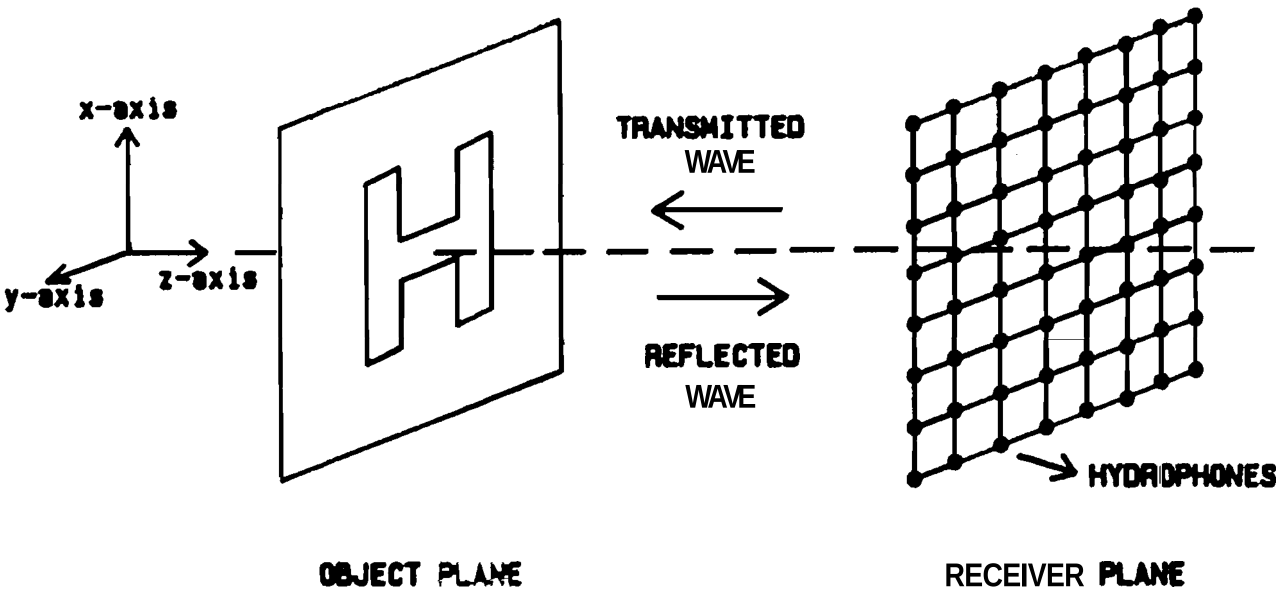


Fig.1.1 A typical acoustic imaging setup.

Acoustic waves are transmitted from one end. They hit the target, and the reflected data is received by an array of hydrophones. This data is converted to electric signals, and processed to form an image of the object. The coordinates on the object plane are referred to as $x_o$, $y_o$, and $z_o$, whereas those on the object plane are referred to as x, y, and z.

on the received data before image formation. This way, the quality of the image can be improved significantly.

In the acoustic holographic technique the receiver is an array of hydrophones. The acoustic field data is received at this hydrophone array. Both the magnitude and phase of the field are measured. These measurements are then converted to electrical signals using suitable transducers. These electrical signals are numeric representation of the data. Their phase and magnitude represent the phase and magnitude values of the received data. The circuits required to measure the data, its conversion to electrical signals, and the subsequent processing to form the image, are all quite complex [2]. The data in acoustic imaging is partial because:

(1) The number of hydrophones on the receiver array is limited. Therefore the measured data is a finite array of complex numbers.

(2) Both the phase and magnitude information may not be available for reconstruction. In particular, therefore, we consider reconstruction from full or quantized phase information of the data.

(3) The data values are corrupted by channel or circuit noise.

We demonstrate that the algorithms proposed in this thesis may solve some of the problems encountered in acoustic imaging.

## 1.3 THEORY OF IMAGE FORMATION

The scalar diffraction theory of light forms the heart of the theory of acoustic holography [8]-[10]. An aperture, when illuminated by a plane wavefront, causes a diffraction pattern to appear at a screen or receiver plane kept at a certain distance from it. The diffraction pattern on the receiver plane can be found by solving the wave equation, if we know the field distribution of the illuminating wave at the aperture plane. General solution to the wave equation is difficult to obtain. However, when subject to certain simplifying assumptions, these equations are more easily solved.

The relation between the received data at $z=z$ and the field on the object plane at $z=z_0$ is given by the Rayleigh - Sommerfield integral [3]

$$g(x,y,z) = A \iint_{x_0\, y_0} g_0(x_0,y_0,z_0)\ \exp\,(-j\ \mathbf{k}.\mathbf{r})\ dx_0\ dy_0 \tag{1.1}$$

where

| | |
|---|---|
| $g_0(x_0, y_0)$ | : acoustic field on the object plane |
| $g(x, Y)$ | : acoustic field on the receiver plane |
| r | : vector distance between the object and the receiver elements. |
| **k** | : wavenumber |
| A | : A constant. We will omit this constant throughout simulation studies reported in this work. |

When the object is far from the receiver end, this equation can be written as (omitting the constant A).

$$g(x,y,z) = \iint_{x_o\,y_o} g_o(x_o,y_o,z_o)\ f(x,x_o,y,y_o,z,z_o)\ dx_o\ dy_o \tag{1.2}$$

where

$$f(x,x_o,y,y_o,z,z_o) = \exp\{-j\frac{2\pi z}{\lambda}[(x-x_o)^2+(y-y_o)^2+(z-z_o)^2]\}$$

We can also write

$$g(x,y,z) = g_o(x,y,z) * h(x,y,z) \tag{1.3}$$

By taking the Fourier transform of both sides of (1.3) we get

$$G(f_x,f_y,z) = G_o(f_x,f_y,z) \cdot H(f_x,f_y,z) \tag{1.4}$$

where $h(f_x,f_y,z) = \exp\{-j\frac{2\pi z}{\lambda}[1+(1-\lambda f_x)^2+(1-\lambda f_Y)^2]\}$

$G_o$ can be obtained from G by

$$G(f_x,f_y,z) = G_o(f_x,f_y,z) \cdot H^{-1}(f_x,f_y,z) \tag{1.5}$$

Now $g_o(x_o,y_o,z_o)$ can be computed from $G_o(f_x,f_y,z)$ by taking the inverse Fourier transform. We have followed this procedure in our simulation studies.

The work reported in this thesis has been done to reduce the circuit and measurement complexity in the problems of signal reconstruction from data collected by an array of receivers. We report results on

(1) The importance and use of the quantized phase information to reduce the circuit complexity.

(2) The use of quantized phase measurements made at multiple frequencies to reduce the receiver array size.

(3) The use of quantized phase information to reduce the effects of noise in the measured data.

We have concentrated on signal reconstruction from the phase of the received data. As the results developed in this thesis will show, the phase of the received data can be manipulated to improve the image clarity and to reduce the receiver array complexity. The phase is one of the measurements made anyway, therefore we do not have to modify the measurement procedure.

The thesis is organized as follows. In chapter 2 we review the work reported in literature related to reconstruction of signals from the phase information of their Fourier transforms. We will also study its applicability to the class of signals under consideration in this work. In chapter 3 we study the use of the quantized phase information for image reconstruction. Recently a technique of signal reconstruction using multiple frequencies has been proposed. In chapter 4 we study the application of this technique to the reconstruction of images from quantized phase information of the received data. In chapter 5 we show that the quantized phase information helps to reduce the effects of noise in the measured data.

# CHAPTER 2

# SIGNAL RECONSTRUCTION FROM PHASE OF RECEIVED DATA

## 2.1 SIGNAL RECONSTRUCTION AS A PARTIAL DATA PROBLEM

In all problems of signal reconstruction from array data, the data collected is usually a set of complex numbers. At each data point, we have a phase and a magnitude value. Ideally, the phase and magnitude values of all the points are required for image formation [11],[12]. But due to certain factors like measurement errors, noise, etc., only the phase or only the magnitude information may be available. This partially available information must be used to recover the original signal. It is similar to other signal recovery problems discussed in literature [13]-[15].

Significant work has been done on the possibility of signal reconstruction from only the Fourier transform phase or only the Fourier transform magnitude information [16]-[21]. Conditions have been stated in literature under which it is possible to recover a signal from only one of the above information or from a mixture of the two. The signal reconstruction from acoustic field data is different from the reconstruction from the Fourier transform in the standard image processing.

In this chapter we concentrate on the conditions for signal recovery from the phase of the received data. Technique of signal reconstruction from phase is attractive as it can be used to reduce the measurement complexity also.

At each receiver element, only the phase measurement would be required, and therefore the magnitude measurement can be avoided.

In section 2.2 we state the conditions under which a signal can be recovered from only its Fourier transform phase information. The results stated in literature place certain constraints on the signal for its recovery from only the Fourier transform phase information. In section 2.3 we show that some additional information in the signal domain can help to relax these constraints. In section 2.4 we present an algorithm for signal reconstruction from phase data. Various iterative and non-iterative algorithms have been proposed for signal recovery from partial data [22]-[25]. The POCS (Projections Onto Convex Sets) algorithm [26]-[29] has been used in this work. It is an iterative algorithm and we found it suitable for our work.

## 2.2 TECHNIQUES FOR SIGNAL RECOVERY FROM PHASE

Every signal has a unique Fourier transform. Therefore a signal is completely specified by its Fourier transform. Given the complete Fourier transform information, the original signal can be recovered uniquely. But when the Fourier transform information is not known completely, it is not always possible to recover the original signal. In this section we study the problem of signal recovery from only the Fourier transform phase information. A primary result for signal recovery from only the Fourier transform phase information can be stated as follows [16]:

**Theorem** 2.1 : Let $x(n)$ be a real one-dimensional sequence which is zero outside the interval $0 \le n \le N-1$ with $x[0] \neq 0$ such that its z-transform does not have any zeros in reciprocal pairs. Let $y(n)$ be another sequence which is zero outside the interval $0 \le n \le N-1$. Let $\theta_x(f)$ and $\theta_y(f)$ be the Fourier transform phase functions for $x(n)$ and $y(n)$ respectively. If $\theta_x(f) = \theta_y(f)$ at $N-1$ distinct frequencies in the interval $0 < f < \pi$, then $y(n) = a\, x(n)$ for some positive constant a. If $\tan\theta_x(f) = \tan\theta_y(f)$ at $N-1$ distinct frequencies in the interval $0 < f < \pi$, then $y(n) = b\, x(n)$ for some real constant b.

We will not repeat the proof of this theorem here. But we show why there is a restriction on the presence of reciprocal zeros in the z-transform of the signal. The zeros in the z-transform of a real sequence $x(n)$ occur in complex conjugate pairs. Thus, if there is a zero at $z_o$, there will also be a zero at the complex conjugate $z_o^*$. In addition, if there is a zero at $1/z_o$, there will be another zero at $1/z_o^*$. Thus a part of the z-transform, $X(z)$, of the signal will be

$$(z-z_o)(z-z_o^*)(z^{-1}-z_o)(z^{-1}-z_o^*) \qquad (2.1)$$

These four terms together give a real quantity. Therefore, this set of four zeros adds only zero or $\pi$, uniformly, to the overall phase of $X(f)$ (Fourier transform of $x(n)$). In either case, it is not possible to detect the presence of these four zeros by the knowledge of the Fourier transform phase alone. Therefore, it is not possible to recover the original signal in such situations.

This theorem states that if the z-transform of a real sequence does not have any zeros in reciprocal pairs, it is possible to recover it from its Fourier transform phase information alone. Group-delay functions can be used to explain the same concept in an elegant way [30],[31].The standard POCS algorithms can be used for this reconstruction [27].

---

(1) Pick any real sequence as the initial estimate of $x(n)$.

**repeat**

(2) compute $X(f)$. /* the Fourier transform of $x(n)$ */

(3) apply phase correction at the points where Fourier transform phase values of $X(f)$ are known. This gives the next refined estimate of $X(f)$.

(4) Compute the next estimate of $x(n)$ by taking the inverse Fourier transform of $X(f)$. Apply finite support constraint on $x(n)$.

**until** an acceptable solution is achieved.

(5) Stop.

Algorithm 2.1 An algorithm for signal recovery from the Fourier transform phase information.

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Fig.2.1(b) shows the signal recovered from only the Fourier transform phase information for the original real sequence shown in Fig.2.1(a). Algorithm 2.1 was used for signal recovery. Fig.2.2(b) shows the result obtained from only the Fourier transform phase information for a real sequence shown in Fig.2.2(a). The sequence in Fig.2.2(a) has

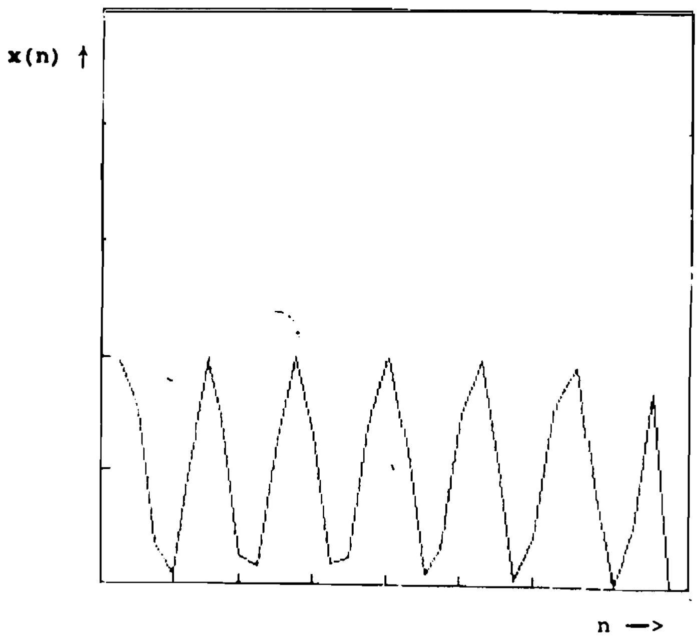



**Fig.2.1(a)** A one-dimensional signal used to study the reconstruction from Fourier transform phase information.

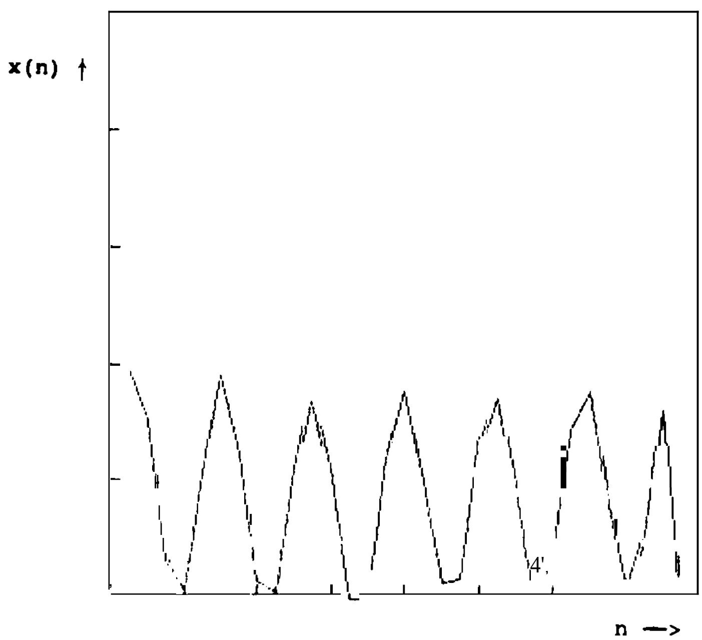



**Fig.2.1(b)** Signal recovered from only the phase of the Fourier transform of the signal in Fig.2.1(a).

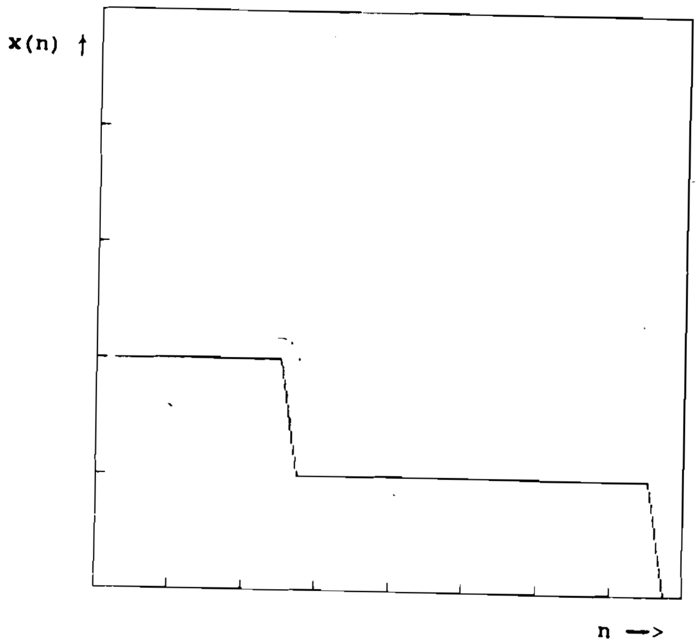



**Fig.2.2(a)** A one-dimensional real sequence whose z-transform has a pair of reciprocal zeros.

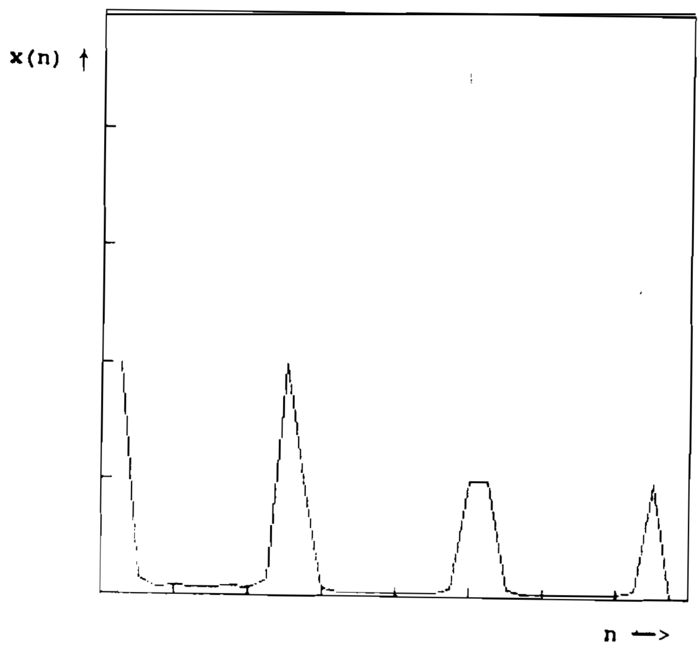



**Fig.2.2(b)** The signal recovered from only the Fourier transform phase of the signal in **Fig.2.2(a)**. Since the **z-transform** of the original signal has a pair of reciprocal zeros, it is not possible to recover it from the Fourier transform phase **information** alone.

a pair of reciprocal zeros. It can be seen that in this case only the Fourier transform phase information is not sufficient to recover the original signal.

Theorem 2.1 states the conditions for the reconstruction of a one-dimensional signal from only the Fourier transform phase information. The corresponding result for two-dimensional signals can be stated as follows [17]:

Theorem 2.2 : Let $x(n,m)$ and $y(n,m)$ be two real two-dimensional sequences with region of support which is zero outside the intervals $0 \le n \le N-1$ and $0 \le m \le M-1$. If $X(z_1,z_2)$ and $Y(z_1,z_2)$ have no non-trivial symmetric factors, and $\theta_x(f_1,f_2) = \theta_y(f_1,f_2)$ for all frequencies, then $x(n,m) = a\, y(n,m)$ for some positive real number a. If $\tan[\theta_x(f_1,f_2)] = \tan[\theta_y(f_1,f_2)]$ for all frequencies, then $x(n,m) = b\, y(n,m)$ for some real constant b.

The z-transform of a sequence $x(n_1,n_2)$ is defined to be ~~is said to have symmetric factors if~~ symmetric if there are positive integers k & l such that ~~there are two factors $A(z_1,z_2)$ and $B(z_1,z_2)$ such that~~

$$X(z_1,z_2) = \pm z_1^{-k} z_2^{-l} X(z_1^{-1},z_2^{-1})$$

We see that the conditions for the sequences to be reconstructible from their Fourier transform phase information are similar in both the one-dimensional and two-dimensional cases. The main constraint is that there should be no non-trivial symmetric factors in the z-transform of the sequence. In the next section we show that this constraint can be relaxed if some signal samples are known.

## 2.3 RECONSTRUCTION FROM PHASE WITH ADDITIVE SIGNAL INFORMATION

In this section we show that with the knowledge of some signal domain information, it is possible to recover signals from their Fourier transform phase information even when their z-transforms have symmetric factors. The result for the one-dimensional case can be stated as follows [32]:

**Theorem 2.3 :** Let $x(n)$ be a real sequence which is zero outside the interval $0 \leq n \leq N-1$ with $x[0] \neq 0$. Assume that the z-transform of $x(n)$ has one pair of reciprocal zeros. Let $y(n)$ be any real sequence which is zero outside the interval $0 \leq n \leq N-1$. If $\theta_Y(f) = \theta_x(f)$ at N-5 distinct frequencies in the interval $0 < f < \pi$, and $y[n] = x[n]$ for the first three values of n, then $y(n) = x(n)$.

Proof : If the z-transform of a real, finite-duration sequence $x(n)$, has a pair of reciprocal zeros, then it can be written as:

$$X(z) = X_1(z).X_2(z) \tag{2.2}$$

where $X_1(z)$ has no zeros in reciprocal pairs and

$$X_2(z) = (z^{-1}-z_o)(z^{-1}-z_o^*)(z-z_o)(z-z_o^*) \tag{2.3}$$

If $X_2(f)$ is the Fourier transform corresponding to $X_2(z)$, then, as mentioned earlier, $X_2(f)$ adds either 0 or $\pi$, uniformly, to the phase of $X(f)$. This is because $X_2(z)$ has two zeros in reciprocal pairs and and two more that are their conjugates. Therefore, when the z-transform is evaluated on the unit circle to get the Fourier transform of $x_2(n)$, we get a real, even sequence. $x_2(n)$ is a 5-point real, even sequence and $x_1(n)$ is an (N-4)-point real sequence. For the time being let us consider the case when the phase added due to $X_2(f)$ is 0. Later we will show that the results hold even if the phase added is $\pi$. From (2.2) we notice that

$$x(n) = x_1(n) * X_2(n) \tag{2.4}$$

where $x_1(n) \leftrightarrow X_1(z)$ and $x_2(n) \leftrightarrow X_2(z)$

Therefore $x(n)$ is formed from convolution of an $(N-4)$-point real sequence with a 5-point real, even sequence. $x_1(n)$ can be determined to within a scale factor by knowing $(N-5)$ distinct phase values in the range $0<f<\pi$ (Theorem 2.1). Since the Fourier transform phase of $x_1(n)$ is equal to that of $x(n)$, these $(N-5)$ phase values can be obtained from the phase of $X(f)$. If required, the scale factor can also be determined by knowledge of at least one value of $x_1(n)$.

To prove the theorem, we have to show that if $x(n)$ can be completely recovered from the knowledge of $x_1(n)$ and $x_2(n)$, then it can also be fully determined with the knowledge of $x_1(n)$ and the first three samples of $x(n)$.

The convolution equation (2.4) can be written as:

$$\begin{aligned}
x_1[0].x_2[0] &= x[0] \\
x_1[0].x_2[1] + x_1[1].x_2[0] &= x[1] \\
x_1[0].x_2[2] + x_1[1].x_2[1] + x_1[2].x_2[0] &= x[2] \\
\vdots \qquad &\quad \vdots \\
x_1[N-6].x_2[4] + x_1[N-5].x_2[3] &= x[N-2] \\
x_1[N-5].x_2[4] &= x[N-1]
\end{aligned} \tag{2.5}$$

$x_1(n)$ can be determined by the use of Theorem 2.1. $x(n)$ can be computed with the knowledge of $x_1(n)$ and $x_2(n)$ using the set of equations 2.5. Since $x_2(n)$ is a 5-point real, even sequence, only three independent values of $x_2(n)$ are enough to specify it uniquely. Suppose that $x_2(n)$ is not known but the first three samples of $x(n)$, i.e, $x[0]$, $x[1]$, and $x[2]$, are available. Then, by using the first three equations in (2.5), $x_2(n)$ can be determined. Now since both $x_1(n)$ and $x_2(n)$ are available, the rest of $x(n)$ can be computed.

Thus we have shown that if $x_1(n)$ can be computed independently, the required additional information about $x_2(n)$ is equivalent to knowing the first three samples of $x(n)$. We have already shown that $x_1(n)$ can be determined from the phase information of $X(f)$. Furthermore, in this case, even without knowing any values in the sequence $x_1(n)$, $x(n)$ will be computed to the correct scale factor. If, by the use of Theorem 2.1, $x_1(n)$ is determined as, say $ax_1(n)$ for a positive constant a, then when equation 2.5 is used, $x_2(n)$ will be determined as $(1/a)x_2(n)$, since $x[0]$, $x[1]$, and $x[2]$ are known completely. Thus the rest of the $x(n)$ values will also be known to the correct scale factor. Now we consider the case when the phase added due to $X_2(f)$ is $\pi$. When this is so, $x_1(n)$ will actually be determined as $-ax_1(n)$ for some constant a. Then $x_2(n)$ will be determined as $-(1/a)x_2(n)$ when the set of equations 2.5 is used for reconstruction. Since $x(n)$ is a convolution of the two, it will be recovered correctly. Hence the proof of the theorem is complete.

In the proof of this theorem we have assumed that the first three values of $x(n)$ are known. The knowledge of three arbitrarily chosen samples may not be sufficient. The known samples of $x(n)$ should be such that the corresponding equations in the set (2.5) are independent. Randomly chosen three samples of $x(n)$ may not give independent equations for complete recovery of $x_2(n)$.

We illustrate these results with the use of the signal $x(n)$ in Fig.2.3(a). The z-transform of this signal has a pair of reciprocal zeros. Fig.2.3(b) shows the signal reconstructed from the Fourier transform phase information alone. As expected, only the Fourier transform phase information is not sufficient to recover $x(n)$. Figs.2.3(a) and 2.3(b) are identical to Figs.2.2(a) and 2.2(b), respectively. They are reproduced for the convenience of comparison with the following results. Fig.2.3(c) shows the signal recovered from the Fourier transform phase information and the knowledge of the first three signal samples. This signal bears a close resemblence to the original signal. Fig.2.3(d) shows the reconstructed signal when the three known samples do not give independent equations for the recovery of $x_2(n)$. It is not possible to reconstruct the original signal from this information. This illustrates the various aspects of Theorem 2.3.

Theorem 2.3 states the conditions under which a sequence whose z-transform has one pair of reciprocal zeros can be determined from the phase of its Fourier transform. The result can be extended to sequences whose z-transforms have more pairs of reciprocal zeros. Theorem 2.4 states these conditions.

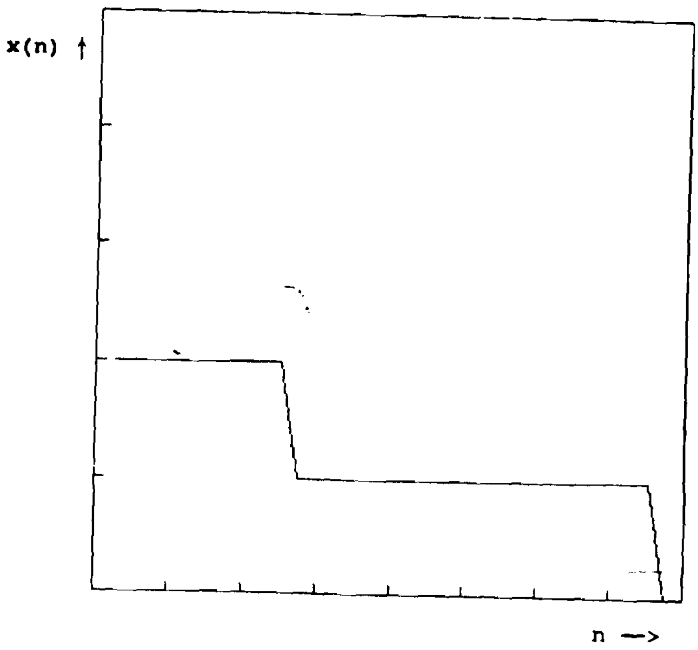



**Fig.2.3(a)** A one-dimensional real sequence whose z-transform has a pair of reciprocal zeros.

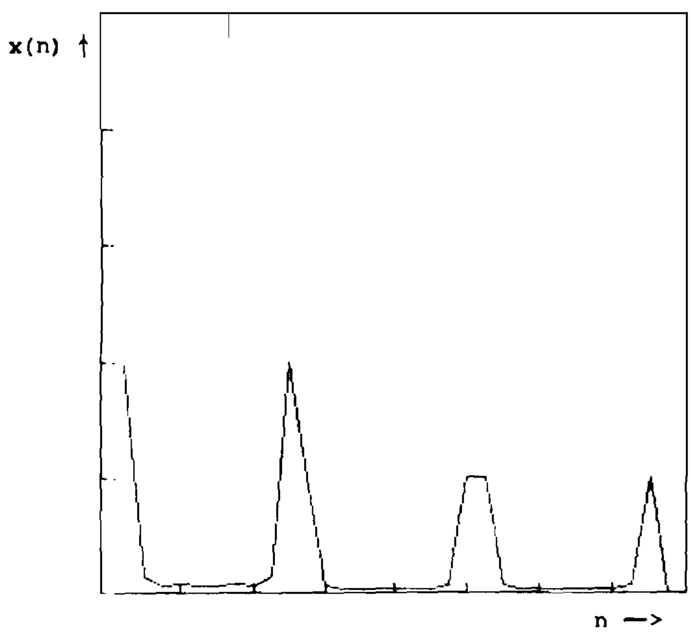



**Fig.2.3(b)** The signal recovered from only the Fourier transform phase of the signal in **Fig.2.3(a).** Since the z-transform of the original signal has a pair of reciprocal zeros, it is not possible to recover it from the Fourier transform phase information alone.

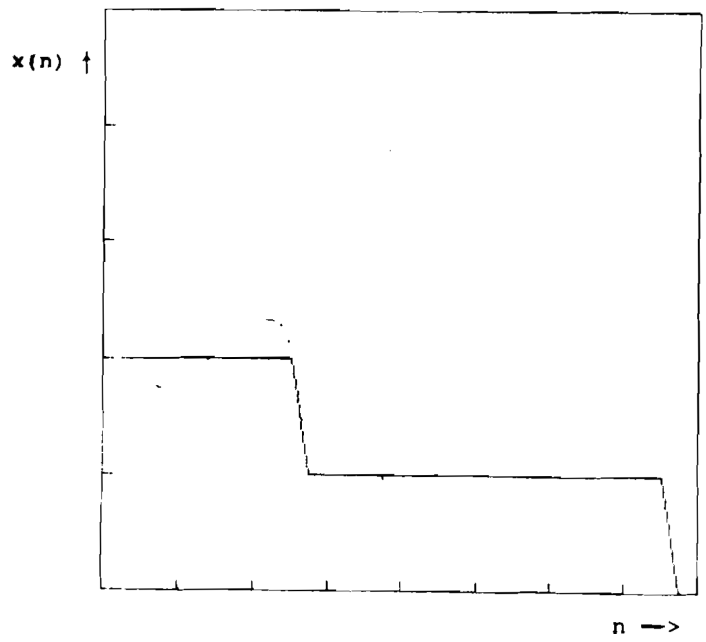



**Fig.2.3(c)** The signal recovered from the Fourier transform phase and the knowledge of the first three samples of the signal in Fig.2.3(a). Complete recovery is possible in this case.

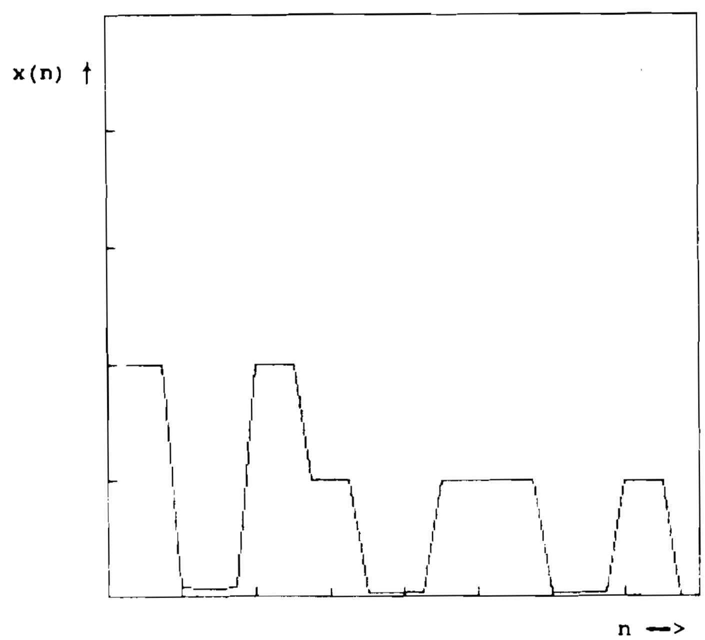



**Fig.2.3(d)** The signal recovered from the Fourier transform phase and three samples of the signal in Fig.2.3(a). These three samples do not give independent equations for signal recovery (Theorem 2.3). Therefore, it is not possible to recover the original signal.

Theorem 2.4 : Let $x(n)$ be a real sequence which is zero outside the interval $0 \leq n \leq N-1$ with $x[0] \neq 0$. Let the z-transform of $x(n)$ have m pairs of reciprocal zeros. Let $y(n)$ be any real sequence which is zero outside the interval $0 \leq n \leq N-1$. If $\theta_y(f) = \theta_x(f)$ at $(N-(4m+1))$ distinct frequencies in the interval $0 < f < \pi$, and $y[n] = x[n]$ for first $\lceil (4m+1)/2 \rceil$ ($\lceil x \rceil$ stands for the smallest integer greater than or equal to x) distinct values of n, then $y(n) = x(n)$.

Proof : It is a straightforward extension of Theorem 2.3.

It may be noted here that if the number of known samples of $x(n)$ is less than the minimum specified in Theorems 2.3 and 2.4, the sequence $x_2(n)$ cannot be determined, no matter how many phase samples we have in the interval $0<f<\pi$. Then it will not be possible to reconstruct the original sequence. Although this result has been developed only for a one-dimensional sequence here, we can show that two-dimensional sequences which have a few symmetric factors can be recovered from their Fourier transform phase information and knowledge of a few signal samples.

In sections 2.1 and 2.2 we have listed the conditions under which a real one-dimensional sequence can be reconstructed from the phase of its Fourier transform. The signals considered in these sections were real-valued. In the next section we study the applicability of these results for complex-valued signals because these are the kind of signals we deal with in our applications.

## 2.4 IMAGE RECONSTRUCTION FROM PHASE DATA

It is .possibleto recover a complex-valued signal from only the phase of its Fourier transform. This result can be developed quite easily for two-dimensional signals. We state a well known theorem from algebra that will be used to develop this result and will also be used in the next chapter for developing the results on signal reconstruction from quantized Fourier transform phase information.

**Theorem 2.5** : [33] If $X(z_1,z_2)$ and $Y(z_1,z_2)$ are two-dimensional polynomials of degrees r and s with no common factors of degree > 0, then there are at most **r*s** (r multiplied by s) distinct pairs $(z_1,z_2)$ where

$$X(z_1,z_2) = 0$$

and

$$Y(z_1,z_2) = 0.$$

The degree of a polynomial of two variables, $X(u,v)$, is defined as the **max(power(u)+power(v))** in the polynomial. For example, the degree of the polynomial $P(u,v) = uv + u^2v^5 + u^3v^3$, is 7 because of the term $u^2v^5$, and in this term the sum of the degrees of u and v is 7. The sum of the degrees of u and v in all other terms is less than 7.

The zero-crossing points of two-dimensional polynomials fall on some contours on the $(u,v)$ plane. This theorem states that if two two-dimensional polynomials of degree r and s do not have any common factors, then their zero-crossing contours cannot intersect in more than **r*s** points. This can be used for developing some important results. According to this theorem, it is not possible for two distinct two-dimensional polynomials, each of degree r, to have more than $r^2$ common zero-crossing points. Therefore if it is known that two irreducible two-dimensional polynomials of degree at most r, have more than $r^2$ common zero-crossing points, then the two can differ only by a scale factor. Otherwise Theorem 2.5 will be contradicted. This theorem can be used to prove the following result which will be of use to us.

**Theorem 2.6** : Let $x(n_1,n_2)$ and $y(n_1,n_2)$ be two-dimensional sequences which are zero outside the rectangle bounded by $0 \leq n_1,n_2 \leq N$. If the z-transforms of the even and odd parts of both $x(n_1,n_2)$ and $y(n_1,n_2)$ are irreducible, $Re(x[0,0]) \neq 0$ and $Im(x[0,0]) \neq 0$, and $phase\{Y(f_1,f_2)\} = phase\{X(f_1,f_2)\}$ at all frequencies, and if $y[0,0] = x[0,0]$, then $x(n_1,n_2) = y(n_1,n_2)$.

**Proof** : A complex two-dimensional sequence can be written as a sum of an even and an odd two-dimensional sequences. Let $x(n_1,n_2)$ be a complex two-dimensional sequence. Then

$$x(n_1,n_2) = x_e(n_1,n_2) + x_o(n_1,n_2)$$

Taking the Fourier transform of both side we have

$$X(f_1,f_2) = X_e(f_1,f_2) + X_o(f_1,f_2)$$

We also know that

$$X_e(f_1,f_2) = Re(X(f_1,f_2))$$

and

$$X_o(f_1,f_2) = j.Im(X(f_1,f_2))$$

where $X_e(f_1,f_2)$ and $X_o(f_1,f_2)$ are the Fourier transforms of $x_e(n_1,n_2)$ and $x_o(n_1,n_2)$, respectively.

If the phase of $X(f_1,f_2)$ is known at all frequencies, then the zero crossings of the $Re(X(f_1,f_2))$ and $Im(X(f_1,f_2))$ are also known. $Re(X(f_1,f_2))$ has zero-crossing points whenever the phase crosses the $\phi = (2n+1)\pi/2$ lines and $Im(X(f_1,f_2))$ has zero-crossing points when the phase crosses the $\phi = \pi n$ lines. Using Theorem 2.5, we see that if the z-transform of a two-dimensional sequence is irreducible, then the two-dimensional sequence is completely determined from the zero crossings of the z-transform. Therefore, if the z-transforms of $x_e(n_1,n_2)$ and $x_o(n_1,n_2)$ are irreducible, then they can also be recovered from the zero crossings of their z-transforms. Fourier transform of a two-dimensional sequence is obtained by evaluating the z-transform along the contours $|z_1|=1$ and $|z_2|=1$. Therefore the sequence can be determined if a sufficient number of zero-crossing points can be found in its Fourier transform. The number of zero-crossing points required is related to the degree of the z-transform, and hence, to the extent of the finite support the signal is known to have. This means that $X_e(f_1,f_2)$ and $X_o(f_1,f_2)$ can be determined from the zero-crossing points of the real and the imaginary part of the Fourier transform. Fourier transform being a one-to-one relation between time and frequency domain signals, $X_e(f_1,f_2)$ and $X_o(f_1,f_2)$ specify $x_e(n_1,n_2)$ and $x_o(n_1,n_2)$ respectively. But the even and odd components of the reconstructed signal may differ from the even and odd components of the actual signal by some scale factors, which may be different for both of them. The knowledge of $x[0,0]$ and the constraint that both $Re(x[0,0])$ and $Im(x[0,0])$ are nonzero, causes these scale factors to be equal. Then these two components specify $x(n_1,n_2)$ completely.

Theorem 2.6 states that if the z-transforms of the even and odd parts of a complex-valued two-dimensional signal are irreducible, it can be recovered from only its Fourier transform phase information. The steps involved for image formation in acoustic imaging are outlined in Section 1.3. Two Fourier transforms have to be computed in the process, and it also involves multiplication by a complex-valued factor. Therefore, the technique of image formation is not a simple Fourier transforming in our case. But as the data in acoustic imaging is complex-valued, if the conditions laid down in Theorem 2.6 are satisfied, we expect to reconstruct images from the phase of the received data in acoustic imaging. We have found experimentally that a wide range of signals are reconstructible in this manner.

Fig.2.4(a) shows an object chosen for study in this work. Only two-dimensional objects are considered in simulation studies, because it is difficult to simulate the acoustic field data at the object end for general three-dimensional objects. Figs.2.4(b) and 2.4(c) show the images formed from only the phase information of the received data after 5 and 20 iterations, respectively, of the POCS algorithm. This illustrates that in acoustic imaging also, signals can be recovered from only phase information of the received data.

The possibility of image formation from phase of the received data can help to reduce the measurement complexity


**Fig.2.4** Image formation from the phase data in acoustic imaging.

(a) Original object **128x128** points (Receiver array **128x128** points).

(b) Image reconstructed from the full phase data after 5 iterations of the POCS algorithm.

(c) Image reconstructed from the full phase data after **20** iterations.

in an acoustic imaging set up. Usually both the phase and magnitude components are measured at the hydrophone array. This method suggests that the magnitude measurement need not be made at all. The phase measurement alone is sufficient for signal recovery.

## 2.5 SUMMARY

When the z-transform of a one-dimensional or a two-dimensional real valued signal does not contain any symmetric factors, the original signal can be specified to within a scale factor from only the Fourier transform phase information. The scale factor can be determined from the knowledge of one signal sample. In the presence of a few symmetric factors, the knowledge of a few signal samples can help in the signal recovery, as it is not possible to recover the signal from only the Fourier transform phase information. When the signal is complex valued, it can be recovered from the Fourier transform phase information if the z-transforms of its even and odd components are irreducible. These results can be used for signal reconstruction from complex-valued data collected at a receiver array. Since the signals are complex valued, more data samples are required for image formation. But the measurement complexity on the receiver array is decreased as the magnitude measurements need not be made in this case. Even the full phase information is not required for recovery of the original signal. The knowledge of zero-crossing points of the real and the imaginary parts of the received data is sufficient for this purpose. This suggests a phase quantization scheme which we discuss in the next chapter.

# CHAPTER 3

# SIGNAL RECONSTRUCTION FROM QUANTIZED PHASE DATA

## 3.1 MEASUREMENT COMPLEXITY AND QUANTIZED PHASE DATA

One of the issues in the class of imaging problems under consideration is to reduce the measurement complexity of the system. The data at each element consists of a phase and a magnitude component. In Chapter 2 we showed that if the number of data samples is large, it is not necessary to make magnitude measurements. The phase information of the received data alone is sufficient for signal recovery. In this chapter we develop this scheme further and show that it is not necessary for the phase information to be very accurate. We show that the quantized phase information of the received data is sufficient for signal recovery. Some results on this topic have already been reported in literature [34]. We study the application of these results for our case. As we will show, since at each receiver element, only the quantized phase information is required, the phase measurement need not be very accurate. Therefore the measurement complexity can be reduced.

If it were possible to reconstruct a signal from the Fourier transform phase information alone, it would mean that the rest of the information in the Fourier transform domain is redundant. It is not so. In general both the phase and magnitude of the Fourier transform are required for signal reconstruction. Only when the signal satisfies certain constraints, it is possible to recover it from its Fourier transform phase information. In such situations, we require a larger number of samples as compared to the case when reconstruction is done from both the phase and magnitude [16]. In situations where the phase-only reconstruction is possible, the lack of magnitude information at all samples is compensated by a larger number of phase samples. Therefore, while these schemes help to reduce the measurement complexity, the number of elements (hydrophones) on the receiver array must be increased. This is also not desirable as it increases the cost of the imaging setup. There are techniques to overcome this difficulty and they will be discussed in Chapter 4. Here we present the results about signal reconstruction from quantized phase information assuming that there are remedies for any side-effects caused by this scheme.

## 3.2 RECONSTRUCTION FROM QUANTIZED FOURIER TRANSFORM PHASE INFORMATION

In this section we study the problem of signal recovery from the quantized Fourier transform phase information. Some results available in literature are presented first. Before stating the results, we explain a few important terms.

(1) A sequence is said to have the region of support as $R(N)$ if the sequence is zero outside the region $-N \le n_1, n_2 \le N$ [34].

(2) A two-dimensional signal has a region of support over a non-symmetric half-plane (NSHP), if $(n_1,n_2)$ is in the region of support implies that $(-n_1,-n_2)$ is not in the region of

support.

(3) $\mathrm{Sign}\{\mathrm{Re}[X(f_1,f_2)]\}$ and $\mathrm{Sign}\{\mathrm{Im}[X(f_1,f_2)]\}$ represent the signs of the real and imaginary parts, respectively, of the Fourier transform of a sequence $x(n_1,n_2)$.

Now we state the first result:

Theorem 3.1 : Let $x(n_1,n_2)$ and $y(n_1,n_2)$ be real two-dimensional sequences with region of support over a nonsymmetric half-plane with $\mathrm{Sign}\{\mathrm{Re}[X(f_1,f_2)]\} = \mathrm{Sign}\{\mathrm{Re}[Y(f_1,f_2)]\}$. If $\mathrm{Re}\{X(f_1,f_2)\}$ takes on both positive and negative values and $X_e(z_1,z_2)$ and $Y_e(z_1,z_2)$ are non-factorable, then $x(n_1,n_2) = c\, y(n_1,n_2)$. for some positive constant c.

The complete proof of this theorem can be found in [34],[35]. Here we give a brief outline of the proof to help in the development of a few more results.

When a real two-dimensional sequence $x(n_1,n_2)$ has its region of support over an NSHP, it can be specified uniquely by its even component. Since the Fourier transform of the even component of a sequence corresponds to the real part of its Fourier transform, the sequence can be recovered from the real part of its Fourier transform. The sign of the real part of the Fourier transform is known at all $(f_1,f_2)$ pairs. This implies that we know the zero-crossing points of the real part of the Fourier transform of the sequence. Suppose that there are two two-dimensional sequences such that the z-transforms of their even and odd components are irreducible, and have degrees of at most s. If the real parts of their Fourier transforms have more than $s^2$ common zero-crossing points, then the two sequences can differ only by a scale factor. This follows directly from Theorem 2.5. Therefore if the sign of the real part of the Fourier transform of a two-dimensional sequence is known at all the frequencies, then the sequence can be specified to a within scale factor, provided the number of zero-crossing points is more than a certain number. This number is a function of the degree of the z-transform of the sequence [34].

This theorem states the conditions under which certain real valued two-dimensional sequences can be recovered from the sign of the phase of real part of their Fourier transforms. The result is easily extendable to complex signals that have their region of support over an NSHP or signals that are symmetric around the origin. This is because such sequences are completely specified by their even components, and therefore, by the real part of their Fourier transforms. Therefore complex valued sequences with region of support over an NSHP, or signals symmetric around the origin, can be recovered from only the sign information of the real parts of their Fourier transforms, provided the z-transforms of the even components of these sequences are irreducible. Theorem 3.1 can be stated now in this form:

Theorem 3.2 : Let $x(n_1,n_2)$ and $y(n_1,n_2)$ be two-dimensional sequences, of the type mentioned above, with region of support over a nonsymmetric half-plane with $\text{Sign}\{\text{Re}[X(f_1,f_2)]\} = \text{Sign}\{\text{Re}[Y(f_1,f_2)]\}$. If $\text{Re}\{X(f_1,f_2)\}$ takes on both positive and negative values and $X_e(z_1,z_2)$ and

$Y_e(z_1,z_2)$ are non-factorable, then $x(n_1,n_2) = c\ y(n_1,n_2)$ for some positive constant c.

If the original signal is a general complex valued signal, it does not conform to the types specified in Theorems 3.1 and 3.2. We want to establish whether such a signal can be recovered from the quantized phase information as mentioned above. If the signal has a finite support, extending to all the four quadrants, in general it cannot be recovered from the real part of its Fourier transform. But the signal can be broken into an even and an odd component. The Fourier transform of the even component is the real part of the Fourier transform of the original signal, and the Fourier transform of the odd component is the imaginary part of the Fourier transform of the original signal. Keeping this in mind, we state the following theorem:

**Theorem 3.3** : Let $x(n_1,n_2)$ and $y(n_1,n_2)$ be two-dimensional sequences with a finite region of support. Let $\text{sign}\{\text{Re}[X(f_1,f_2)]\}$ be identical to $\text{sign}\{\text{Re}[Y(f_1,f_2)]\}$ and $\text{sign}\{\text{Im}[X(f_1,f_2)]\}$ be identical to $\text{sign}\{\text{Im}[Y(f_1,f_2)]\}$. Assume that $\text{Re}\{X(f_1,f_2)\}$ and $\text{Im}\{X(f_1,f_2)\}$ take on both positive and negative values and $X_e(z_1,z_2)$, $X_o(z_1,z_2)$, $Y_e(z_1,z_2)$ and $Y_o(z_1,z_2)$ are non-factorable. If $\text{Re}\{x[0,0]\}$ and $\text{Im}\{x[0,0]\}$ are non-zero and if $y[0,0] = x[0,0]$ then $x(n_1,n_2) = y(n_1,n_2)$.

**Proof** : A complex two-dimensional sequence $x(n_1,n_2)$ can be written as the sum of a two-dimensional even sequence and a two-dimensional odd sequence. The Fourier transform of $x_e(n_1,n_2)$ is equal to the real part of the Fourier transform of $x(n_1,n_2)$ and the Fourier transform of $x_o(n_1,n_2)$ is equal to the imaginary part of the Fourier transform of $x(n_1,n_2)$. Therefore, if $\mathrm{Re}\{X(f_1,f_2)\}$ and $\mathrm{Im}\{X(f_1,f_2)\}$ can be computed independently, the original signal can be recovered uniquely. If the signs of $\mathrm{Re}\{X(f_1,f_2)\}$ and $\mathrm{Im}\{X(f_1,f_2)\}$ are known at all frequencies, and if $X_e(z_1,z_2)$ and $X_o(z_1,z_2)$ are irreducible, then by Theorem 3.2, $x_e(n_1,n_2)$ and $x_o(n_1,n_2)$ can be determined to within some scale factors. The knowledge that $\mathrm{Re}(x[0,0])$ and $\mathrm{Im}(x[0,0])$ are non-zero, and that $x[0,0] = y[0,0]$ causes these scale factors to be equal. $x(n_1,n_2)$ can now be uniquely determined because its even and odd components are known. This completes the proof of this theorem.

Retaining the sign of the real part of the Fourier transform is equivalent to quantizing the phase to two levels. The phase is quantized according to the following scheme:

$$\phi_q = \begin{cases} 0 & \text{if } -\pi/2 \leq \phi < \pi/2 \\ \pi & \text{if } \pi/2 \leq \phi < 3\pi/2 \end{cases}$$

As shown above the phase is quantized to two levels. Since one bit is sufficient to represent two levels, we call it 1-bit phase information.

Similarly, knowing the signs of the real and the imaginary points of the Fourier transform of a two-dimensional sequence is equivalent to quantizing the phase to four levels (we shall refer to it as 2-bit phase, as two bits are sufficient to represent four levels of quantization). The

2-bit quantization is done according to the following scheme:

$$\phi_q = \begin{cases} \pi/4 & \text{if} \quad 0 \leq \phi < \pi/2 \\ 3\pi/4 & \text{if} \quad \pi/2 \leq \phi < \pi \\ 5\pi/4 & \text{if} \quad \pi \leq \phi < 3\pi/2 \\ 7\pi/4 & \text{if} \quad 3\pi/2 \leq \phi < 2\pi \end{cases}$$

Fig.3.1 shows the geometrical interpretation of the 1-bit phase quantization scheme. 1-bit phase information refers to the situation when, for each complex vector in the Fourier transform domain, we retain the information of the half-plane in which it lies. Similarly 2-bit phase means, for each complex vector, we remember the quadrant in which it lies.

So far we have seen that certain class of signals can be specified from 1-bit or 2-bit phase information of their Fourier transforms. The condition laid down in the theorems was that this quantized phase information should be known at all frequencies, though only a finite number of zero-crossing points are required to recover a signal from its quantized Fourier transform phase information. In practice, we deal with discrete signals and discrete Fourier transforms. Therefore it is not possible to compute the Fourier transform phase at all frequencies. The sampled points themselves do not cover all the zero-crossing points. In such situations we can look for sign changes in values at adjacent points. A change of sign indicates the existence of a zero-crossing point between the two points. The number of sign changes or equivalently, the number of zero-crossing points required to reconstruct a discrete, finite-support sequence from its

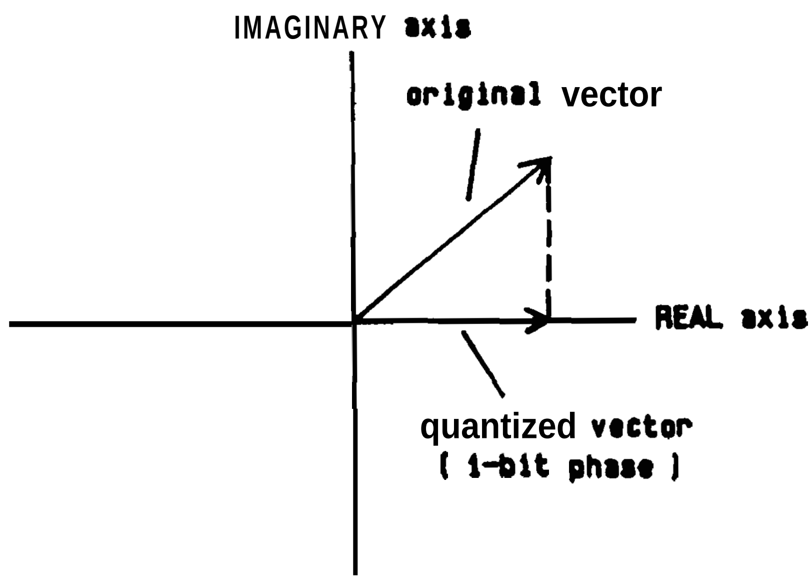



**Fig.3.1** Geometric interpretation of the 1-bit phase quantization scheme. 1-bit phase means retaining the sign of the real part of the complex vector.

quantized Fourier transform phase information is related to the extent of the finite support of the sequence. If there is a two-dimensional sequence with finite support of 4N points (at most 2N in each dimension) then another sequence of the same finite support can have at most $(4N)^2 = 16N^2$ zero-crossing points in common with this sequence. But if we know more than $16N^2$ zero-crossing points of this sequence, then any other sequence with the same finite support and having the same zero-crossing points can differ from it only by a scale factor (Theorem 2.5). Therefore a discrete two-dimensional sequence with a region of support over $R(N)$ is uniquely defined if there are more than $16N^2$ sign changes in the real part of its Fourier transform.

In this section we have seen that two-dimensional real signals or two-dimensional complex signals with region of support over an NSHP can be determined from one-bit phase information of their Fourier transforms. General complex signals with a finite region of support can be determined from two-bit phase of their Fourier transforms. We are now in a position to study the applicability of these results to some practical problems.

### 3.3 ALGORITHM FOR SIGNAL RECOVERY FROM 2-BIT PHASE DATA

In section 3.2 we stated the conditions under which a two-dimensional signal can be recovered from 2-bit phase of the received data. In this section we develop an algorithm to achieve this. The algorithm is based on the POCS technique. The algorithm is presented as applicable for acoustic imaging. The required input is the 2-bit phase information of the received data, the knowledge of the propagation factor $h(x,y)$ and apriori knowledge of the region of support of the original signal. The algorithm is as follows:

---

(1) Let x be any any two-dimensional signal. x serves as the initial estimate of the original signal.

repeat

(2) Impose finite support constraint on x. The resultant signal is the next estimate of the original object. Use this signal to simulate the data at the receiver plane.

(3) Make the 2-bit phase correction in accordance with the POCS algorithm.

(4) Compute the Fourier transform of the sequence.

(5) Multiply the result by the factor $H^{-1}(f_1,f_2)$

(6) Take the inverse Fourier transform of this sequence. The resultant signal x is an estimate of the original object.

until an acceptable solution is obtained

(7) Stop.

Algorithm 3.1 An iterative algorithm based on the POCS technique for image formation from the 2-bit phase data in acoustic imaging.

---

The algorithm given above is based on the POCS technique. It can be proved that if there is a unique solution satisfying the given constraints, then the algorithm converges to it. If the solution set consists of more than one signal, then the algorithm converges to one element of the set.

To prove the convergence, we make the following observations:

(1) Signals that have specified 2-bit phase information form a convex set (see Appendix). Therefore the specification of 2-bit phase information of the received data defines a convex set. The step (3) of the algorithm takes the projection of the simulated data onto the convex set defined by the 2-bit phase measured at the hydrophone array.

(2) Signals that have a specified finite support constraint form a convex set (see Appendix). Therefore knowledge of the finite support of the original signal also defines a convex set. After step (3), we have simulated data which has the correct 2-bit phase information. This is used to reconstruct the signal in steps (4), (5), and (6) of the algorithm given above. In accordance with the known information, the finite support constraint is applied to the signal in step (2). This is equivalent to taking the projection onto the convex set defined by the known finite support constraint.

The algorithm works by taking projections onto the two convex sets alternately. If the solution is unique, the algorithm outlined earlier converges to this solution. Otherwise it converges to one of the solutions in the set of all possible solutions, provided this set is not empty. This proves the convergence of the algorithm (see Appendix).

The results developed in this chapter state the conditions under which a signal can be recovered from only 2-bit phase information of the received data in acoustic imaging. We have seen that a signal is uniquely determined by the knowledge of a certain number of zero-crossing points in the Fourier transform domain. But the algorithm used for reconstruction, as given above, does not use the zero-crossing points. We just measure the 2-bit phase at a certain number of points and then replace this measured 2-bit phase at the appropriate points during each iteration. The knowledge of the 2-bit phase does not imply the knowledge of the zero-crossing points also. But if there is a sign change between two adjacent points, it implies that there is a zero-crossing point between the two. If the sampling rate is high, then the adjacent points will be close to each other. Therefore, if it is known that there is a zero-crossing between the two points, due to the sampled points being close to each other, the location of such a zero-crossing will be known with reasonable accuracy. If the number of such sign changes is more than the number of zero-crossing points required for reconstruction, we can expect a good result. The solution set forms a continuous space. Therefore any inaccuracy in the determination of the zero-crossing points causes the solution set to expand. But if the inaccuracy is small, the solution set will also be small. When one of the elements of this set is picked as the probable solution, we can expect it to resemble the original solution in the basic features. Since we deal with image signals, important features like the edge information, uniform regions in the image, etc., are sufficient for recognition of the object. The images formed from the 2-bit phase data reproduce these features. This statement is justified through the experimental results.

## 3.4 SIMULATION STUDIES

We have seen that the quantized phase information is sufficient for reconstruction of the original signal. In this section, we give experimental results corresponding to the theory developed in this chapter. The results have been obtained for a simulated acoustic imaging setup.

Fig.3.2 shows the two-dimensional object used for our study. Figs.3.3(a) and 3.3(b) show the images reconstructed from 2-bit phase and 1-bit phase data, respectively. These images were obtained after one iteration of the POCS algorithm. The corresponding images obtained after 25 iterations of the algorithm are shown in Figs.3.3(c) and 3.3(d). These images were obtained from only the quantized phase information. In other words, the initial estimate was formed from the quantized phase information and a uniform magnitude value. The object and the image size is 128x128 points. If the correct magnitude values are used, we expect a faster convergence of the algorithm. Figs.3.4(a) and 3.4(b) show the images obtained from 2-bit phase and 1-bit phase data, respectively, after one iteration of the algorithm. In this case, the actual magnitude information was used to form the initial estimate. But in the subsequent iterations, only


Fig.3.2 The two-dimensional object used to study the image formation from quantized phase data.

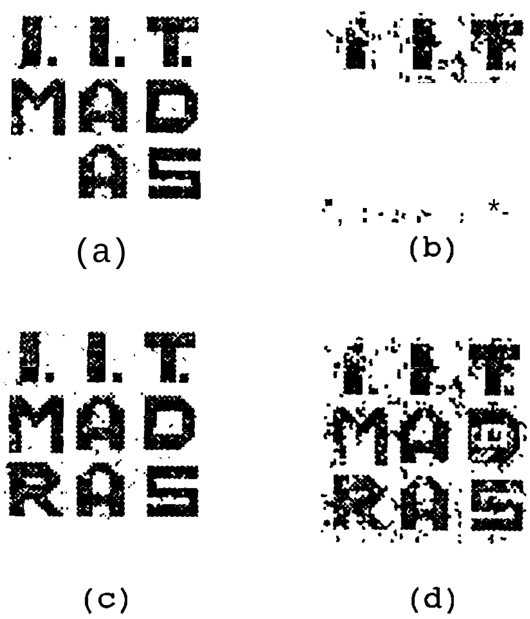


**Fig.3.3** Images reconstructed from quantized (2-bit and 1-bit) phase data.
Original object 128x128 points.
Number of receiver elements 128x128.
Uniform magnitude information used for image reconstruction.
(a) From only the 2-bit phase data (1 iteration).
(b) From only the 1-bit phase data (1 iteration).
(c) From only the 2-bit phase data (25 iterations).
(d) From only the 1-bit phase data (25 iterations).

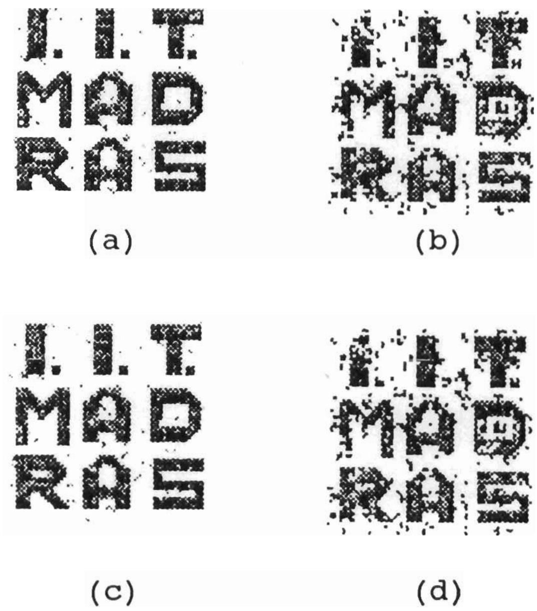



Fig.3.3 Images reconstructed from quantized (2-bit and 1-bit) phase data.
Original object 128x128 points.
Number of receiver elements 128x128.
Uniform magnitude information used for image reconstruction.
(a) From only the 2-bit phase data (1 iteration).
(b) From only the 1-bit phase data (1 iteration).
(c) From only the 2-bit phase data (25 iterations).
(d) From only the 1-bit phase data (25 iterations).

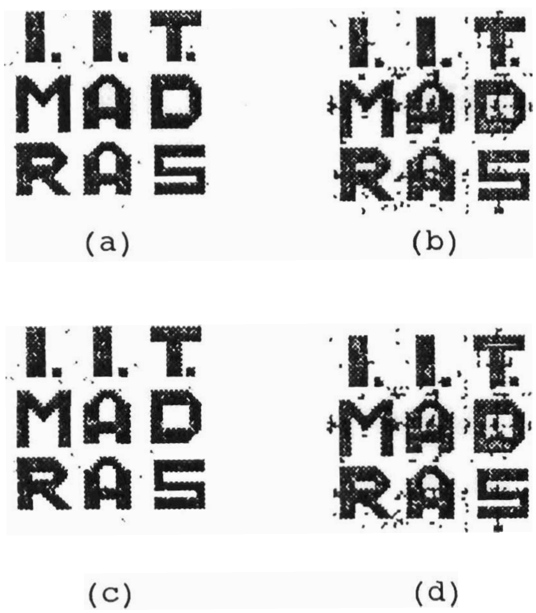



Fig. 3.4 Images reconstructed from quantized (2-bit and 1-bit) phase data.
Original object 128x128 points.
Number of receiver elements 128x128.
Actual magnitude information used to form the initial estimate for image reconstruction.
(a) From the 2-bit phase data (1 iteration).
(b) From the 1-bit phase data (1 iteration).
(c) From the 2-bit phase data (25 iterations).
(d) From the 1-bit phase data (25 iterations).

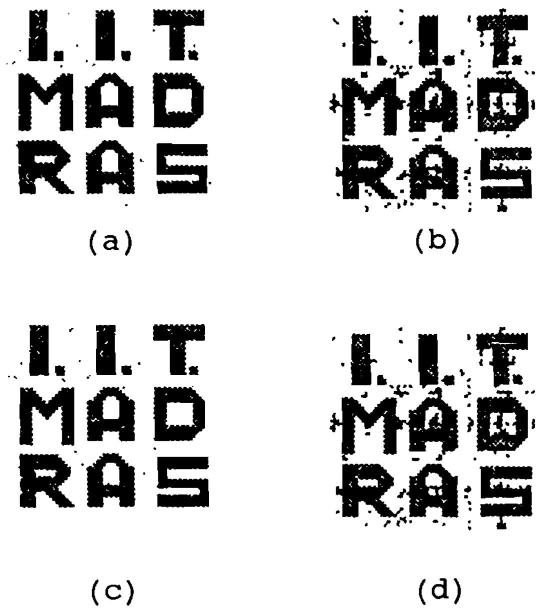



Fig.3.4 Images reconstructed from quantized (2-bit and 1-bit) phase data.
Original object 128x128 points.
Number of receiver elements 128x128.
Actual magnitude information used to form the initial estimate for image reconstruction.
(a) From the 2-bit phase data (1 iteration).
(b) From the 1-bit phase data (1 iteration).
(c) From the 2-bit phase data (25 iterations).
(d) From the 1-bit phase data (25 iterations).

the quantized phase information was used for the data correction. The corresponding imaged obtained after 25 iterations are shown in Figs.3.4(c) and 3.4(d). We observe that retaining the magnitude information helps in a faster convergence to the solution. The convergence of the algorithm to the solution depends on the initial estimate of the signal. Retaining the magnitude of the received data implies that we start with a point closer to the solution than when a uniform magnitude value is assumed for the magnitude. Therefore convergence is faster in the former case.

Though the application of Theorem 3.3 requires the knowledge of $x[0,0]$, we did not use this information for image reconstruction. Experimentally we have found that the finite support constraint is sufficient for image reconstruction.

## 3.5 SUMMARY

A two-dimensional real valued signal can be reconstructed from the 1-bit phase information of its Fourier transform. 1-bit phase information refers to the quantization of phase to two levels. Complex valued signals with a finite support over an NSHP or even signals can also be recovered from the 1-bit phase information. But this result does not hold for the complex valued signals in general. General complex valued signals can be recovered from the 2-bit information of the Fourier transform phase. This refers to the phase quantized to four levels. The 2-bit phase information contains the knowledge of the signs of both the real and imaginary parts of the Fourier transform. In acoustic imaging, an image can be formed from the 2-bit phase information of the received data, since we deal with complex valued signals. We have seen from the experimental results that the 1-bit phase and magnitude information of the received data can also be used to form good images. These techniques require a large number of samples to be available. An algorithm based on the POCS technique can be used for signal recovery from the 2-bit phase information and the knowledge of the finite support of the signal. Phase quantization schemes can be used to reduce the measurement complexity, as the measurements need not be very accurate.

# CHAPTER 4

# SIGNAL RECONSTRUCTION FROM QUANTIZED PHASE AT MULTIPLE FREQUENCIES

## 4.1 RECONSTRUCTION FROM MULTIPLE FREQUENCY DATA

Phase quantization technique helps in the reduction of 'the measurement complexity for the class of imaging problems under consideration in this work. But the number of data samples required to reconstruct the signal from the quantized phase information is more as compared to the case of signal reconstruction from full phase information. To collect the required number of data samples, the receiver array should have a large number of receiver elements. We had started with the aim of reducing the overall complexity, but the need of a large number of receiver elements increases the complexity of the receiver size. Therefore, we must find some way to overcome this difficulty.

Earlier attempts to reduce the receiver array size used synthetic aperture techniques [36]-[38]. But these techniques do not exploit the advantages of iterative algorithms for signal reconstruction from partial information. Our image formation technique is based on the POCS algorithm. In the POCS algorithm, convex sets are formed from the known information about the signal and the collected data. The algorithm converges to one of the elements in the intersection of these convex sets. If this intersection set is small, the solution obtained by using the POCS algorithm is close to the original signal. If this set is large, then the solution may differ from the original signal. But the algorithm ensures that the solution to which it converges is one among the class of signals that satisfy the given constraints. Convergence of the POCS algorithm depends upon the initial estimate of the original signal.

Fig.4.1 illustrates this point. The solution obtained with point a as the initial estimate is different from that obtained with point b as the initial estimate. If the intersection set is small in size, the signal to which the algorithm converges will be close to the original signal. If the available information is accurate, and the amount of information is large, the solution set will be small. But it is not always possible to get more accurate information. Another way to reduce the size of this set is to form a few more convex sets. The solution set will be the intersection of all these sets. As the number of sets increases, the solution set forms a non-increasing sequence in terms of the number of elements. Fig.4.2(a) shows the solution set with only two convex sets. Fig.4.2(b) shows that with the availability of another set, the solution set decreases in size, i.e. the number of elements in the solution set decreases.

Recently a new technique has been proposed for signal reconstruction using data collected at different frequencies [39]-[43]. This technique reduces the size of the intersection of various convex sets, and then makes use of the POCS algorithm to obtain the solution. In section 4.2 we

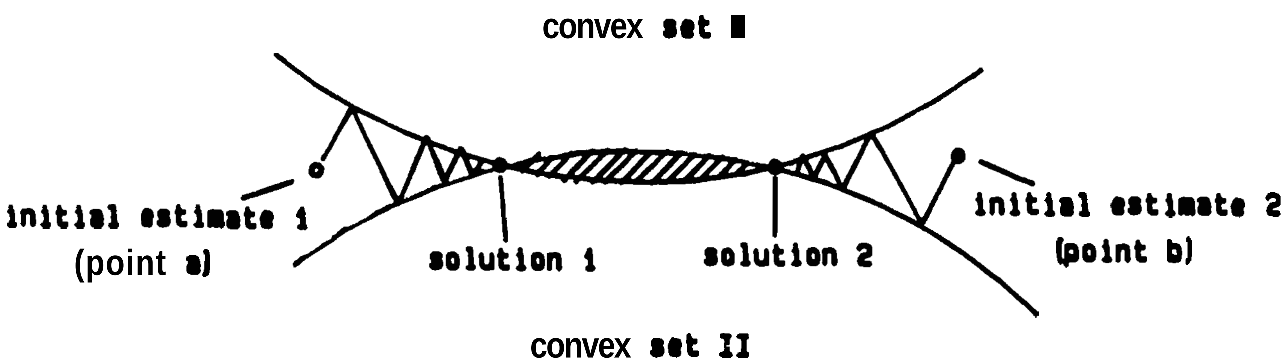



Fig.4.1 The solutions obtained from two different initial estimates may be different. This figure shows two convex sets - set I and set II. The hatched region is the solution set. It can be seen that the solution obtained by choosing point (a) as the initial estimate is different from that obtained by choosing point (b) as the initial estimate.

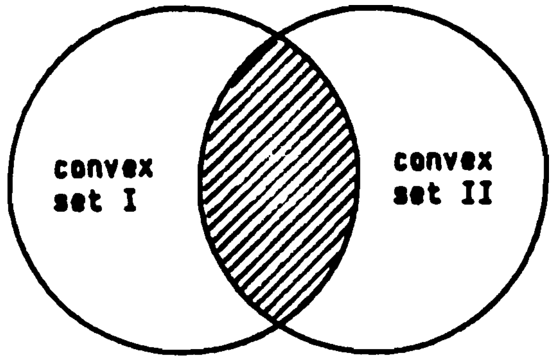



**Fig.4.2(a)** The solution set (hatched region) formed from two convex sets.

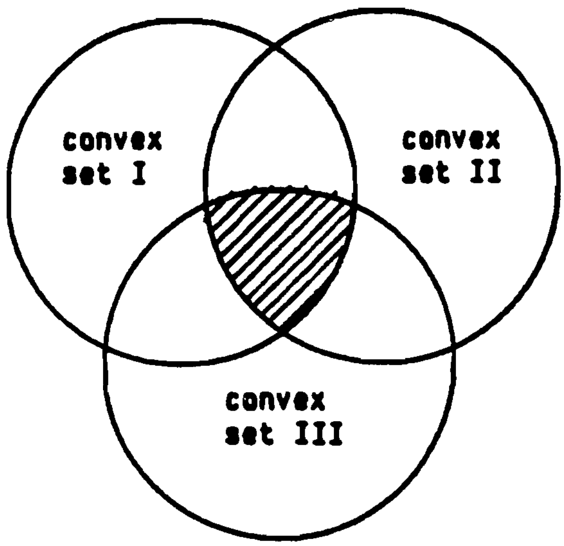



**Fig.4.2(b)** The solution set (hatched region) when one more convex set is added. It can be seen that this set is smaller than (and is a proper subset of) the solution set in Fig.4.2(a).

state this algorithm and outline the main idea behind the technique. The same procedure can be extended to form images with quantized phase data collected at receiver arrays consisting of a small number of receiver elements. The experimental results are discussed in section 4.3.

## 4.2 ITERATIVE ALGORITHM FOR IMAGE RECONSTRUCTION FROM MULTIPLE FREQUENCY DATA

In this section we review the technique of signal reconstruction with the use of multiple frequencies. We state the algorithm used for image reconstruction in acoustic imaging. This algorithm is also based on the POCS technique. We also briefly describe the theoretical aspects of this technique.

It has been proposed that the data collected at multiple frequencies can be used for signal reconstruction in acoustic imaging [43]. Assume that frequencies $f_0$, $f_1, \ldots,$ $f_{N-1}$ are the N frequencies used for data collection. The data is collected by transmitting the wave of each frequency separately and then measuring the field induced at the receiver end due to each of them. This data can then be used in the following algorithm, adapted from [43], to reconstruct the image:

---

**(1)** Take the data collected at frequency $f_0$ as the starting point. Use this data to form the first estimate of the object. Set a variable $i$ to 0.

**repeat**

(2) Increment **i** by 1. Use the estimate of the original signal formed at this stage to simulate the quantized phase data at the receiver end for frequency $f_{i \bmod N}$.

(3) In accordance with the POCS algorithm, correct this simulated quantized phase data with the actually known data samples at the frequency $f_{i \bmod N}$.

(4) Form the next estimate of the original signal from this corrected data.

**until** an acceptable image is formed.

(5) Stop.

<u>Algorithm</u> <u>4.1</u> An algorithm for image reconstruction from the quantized phase data collected at several frequencies.

........................................................................

Discussion of the theory behind signal reconstruction from phase and magnitude information at multiple frequencies can be found in [43]. Here we discuss it briefly because it is useful to develop the results for image formation from the quantized phase information of multiple frequency data.

Equation (1.3) shows that in acoustic imaging the received data can be written as

$$g_f(x,y) = g_o(x,y) * h_f(x,y) \tag{4.1}$$

where $g_o(x,y)$ is the field on the object plane and $h_f(x,y)$ is a factor that arises due to propagation of the acoustic field from the object to the receiver array. This factor is a function of the frequency (f).

For a discrete case it can be written as

$$g_f(n,m) = g_o(n,m) * h_f(n,m) \tag{4.2}$$

or

$$g_f(n,m) = \sum_{n'}\sum_{m'} g_o(n',m').h_f(n-n',m-m') \tag{4.3}$$

The partial data collected at each frequency forms a convex set. Each point in the set represents a possible solution for the data collected at that frequency and all such solutions for the data collected at a particular frequency are contained in the corresponding convex set. The intersection of the various convex sets defines the set of all possible solutions for the data collected at various frequencies. If all the convex sets have only one common point, then this point will correspond to the original signal. In that case, the algorithm 4.1 will converge to it.

The idea behind using several frequencies for signal recovery is that the data collected at each frequency forms a convex set. The set of solutions is the intersection of all such convex sets. By increasing the number of frequencies, we increase the number of convex sets. The intersection of N sets will be a subset of the intersection of a smaller number of these sets. An example will clarify this argument. Let set A be the solution set of the convex sets $C_1$, $C_2$,..., $C_{N-1}$. If some more information is available, and it forms another convex set, the solution set will be the intersection of the sets $C_1$, $C_2$,..., $C_{N-1}$ and $C_N$. It is equal to the intersection of the previous solution set A and the new set $C_N$. Therefore the new solution set is at most as large as the older set A. The solution set is non-empty since we know that the original signal (as yet unknown) satisfies the constraints represented by $C_1$, $C_2, \ldots, C_{N-1}$ and $C_N$. If the additional information is independent of that available previously, the new solution set will be a proper subset of A. The POCS algorithm will converge to an element of this new solution set and it can be expected that the solution thus obtained will be closer to the original solution. For the image signals we are more interested in the object features like edges and uniform regions in the object. Images reconstructed from multiple frequency data reproduce most of these features. Therefore this technique is of practical importance.

The number of frequencies required to make the measurements is dependent on the amount of known data at each frequency. If a small number of samples are known for each frequency, then the number of frequencies required for a complete recovery of the original signal will be larger as compared to the situation when we have comparatively more information at each frequency.

We have seen the basic idea behind the reconstruction of acoustic imaging signals from multiple frequency data for the situation when both magnitude and phase information are available. Similarly, we can expect the images reconstructed from the quantized phase data at multiple frequencies to be better than those reconstructed from the data at a single frequency.

## 4.3 SIMULATION STUDIES

In this section we present the results of experimental studies on acoustic image reconstruction from multiple frequency quantized phase data.

The object shown in Fig.3.2 is used for the experimental studies. Figs.4.3(a) to 4.3(d) show the images formed from the full phase data collected by receiver arrays containing 128x128, 64x64, 32x32, and 16x16 receiver elements. The corresponding images reconstructed from the 2-bit phase data are shown in Figs.4.3(e) to 4.3(h). Those reconstructed from the 1-bit phase data are shown in Figs.4.3(i) to 4.3(1). We see that the image quality decreases if the number of receiver elements on the receiver plane is reduced. The following table gives the list of experiments performed with multiple frequency data and the figures showing the corresponding results.

Table 4.1 describes the studies with full phase information. Tables 4.2 and 4.3 describe the studies with 2-bit phase and 1-bit phase data, respectively.

TABLE 4.1

Results of image reconstruction from the full phase data collected at multiple frequencies.

| IMAGE SIZE | RECEIVER ARRAY SIZE | NUMBER OF FREQUENCIES | RESULT IN FIGURE |
|---|---|---|---|
| 128x128 | 64X64 | 1 | 4.4(b) |
| 128X128 | 64X64 | 2 | 4.4(c) |
| 128X128 | 64X64 | 4 | 4.4(d) |

| | | | |
|---|---|---|---|
| 128X128 | 64X64 | 8 | 4.4(e) |
| 128X128 | 64X64 | 16 | 4.4(f) |
| 128X128 | 32X32 | 1 | 4.5(b) |
| 128x128 | 32X32 | 2 | 4.5(c) |
| 128X128 | 32X32 | 4 | 4.5(d) |
| 128X128 | 32X32 | 8 | 4.5(e) |
| 128X128 | 32X32 | 16 | 4.5(f) |

Figs.4.4(a) and 4.5(a) show the original object.

TABLE 4.2

Results of image reconstruction from the 2-bit phase data collected at multiple frequencies.

| IMAGE SIZE | RECEIVER ARRAY SIZE | NUMBER OF FREQUENCIES | RESULT IN FIGURE |
|---|---|---|---|
| 128X128 | 64X64 | 1 | 4.6(b) |
| 128x128 | 64X64 | 2 | 4.6(c) |
| 128X128 | 64X64 | 4 | 4.6(d) |
| 128X128 | 64X64 | 8 | 4.6(e) |
| 128X128 | 64X64 | 16 | 4.6(f) |
| 128X128 | 32X32 | 1 | 4.7(b) |
| 128X128 | 32X32 | 2 | 4.7(c) |
| 128X128 | 32X32 | 4 | 4.7(d) |
| 128X128 | 32X32 | 8 | 4.7(e) |
| 128X128 | 32X32 | 16 | 4.7(f) |

Figs.4.6(a) and 4.7(a) show the original object.

TABLE 4.3

Results of image reconstruction from the 1-bit phase data collected at multiple frequencies.

| IMAGE SIZE | RECEIVER ARRAY SIZE | NUMBER OF FREQUENCIES | RESULT IN FIGURE |
|---|---|---|---|
| 128x128 | 64X64 | 1 | 4.8(b) |
| 128X128 | 64X64 | 2 | 4.8(c) |
| 128X128 | 64X64 | 4 | 4.8(d) |
| 128X128 | 64X64 | 8 | 4.8(e) |
| 128X128 | 64X64 | 16 | 4.8(f) |
| 128x128 | 32X32 | 1 | 4.9(b) |
| 128X128 | 32X32 | 2 | 4.9(c) |
| 128x128 | 32X32 | 4 | 4.9(d) |
| 128X128 | 32X32 | 8 | 4.9(e) |
| 128X128 | 32X32 | 16 | 4.9(f) |

Figs.4.8(a) and 4.9(a) show the original object.

From the experimental results we make the following observations:

(1) Decreasing the number of receiver elements causes degradation in image quality. This is because the quantized phase data available for reconstruction decreases with decreasing number of receiver elements.

(2) Reduction in the amount of quantized phase data because of the reduced receiver array size can be compensated by increasing the number of frequencies for reconstruction. It can be noticed that as the number of available data points per frequency decreases, a larger number of frequencies are

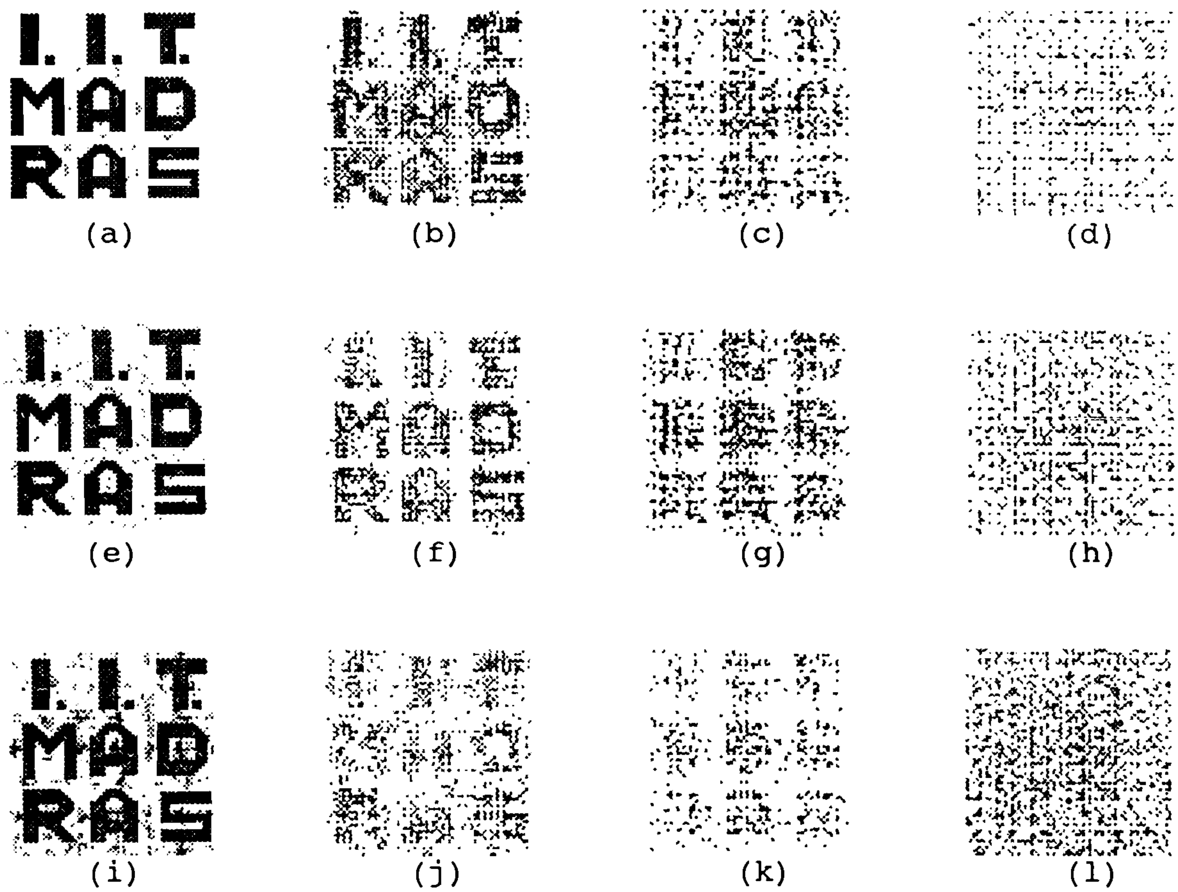



Fig.4.3 Images reconstructed by varying the number of hydrophones on the receiver array.
Original object 128x128 points.

(a),(b),(c),(d) From the full phase data with number of receiver elements 128x128, 64x64, 32x32, and 16x16, respectively.

(e),(f),(g),(h) From the 2-bit phase data with number of receiver elements 128x128, 64x64, 32x32, and 16x16, respectively.

(i),(j),(k),(l) From the 1-bit phase data with number of receiver elements 128x128, 64x64, 32x32, and 16x16, respectively.

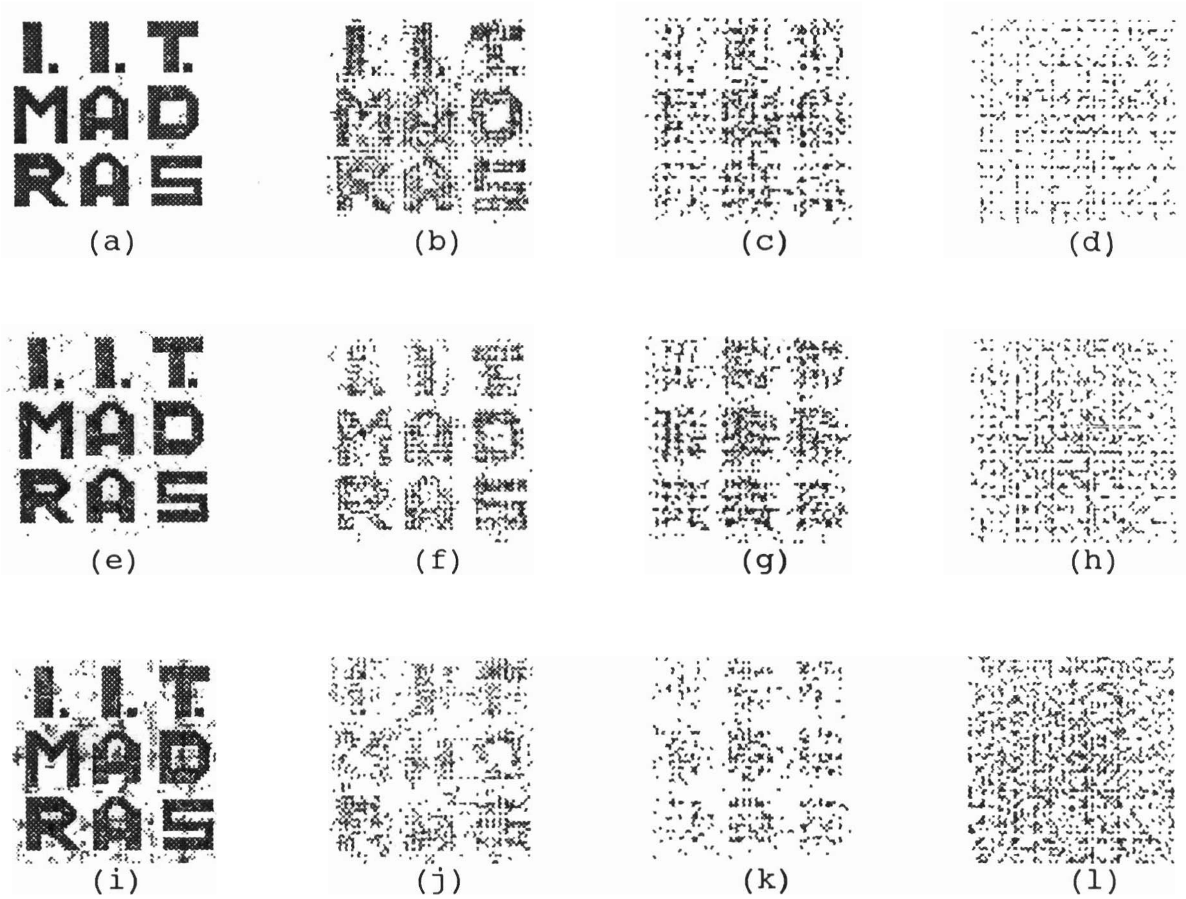



Fig.4.3 Images reconstructed by varying the number of hydrophones on the receiver array.
Original object 128x128 points.

(a),(b),(c),(d) From the full phase data with number of receiver elements 128x128, 64x64, 32x32, and 16x16, respectively.

(e),(f),(g),(h) From the 2-bit phase data with number of receiver elements 128x128, 64x64, 32x32, and 16x16, respectively.

(i),(j),(k),(l) From the 1-bit phase data with number of receiver elements 128x128, 64x64, 32x32, and 16x16, respectively.

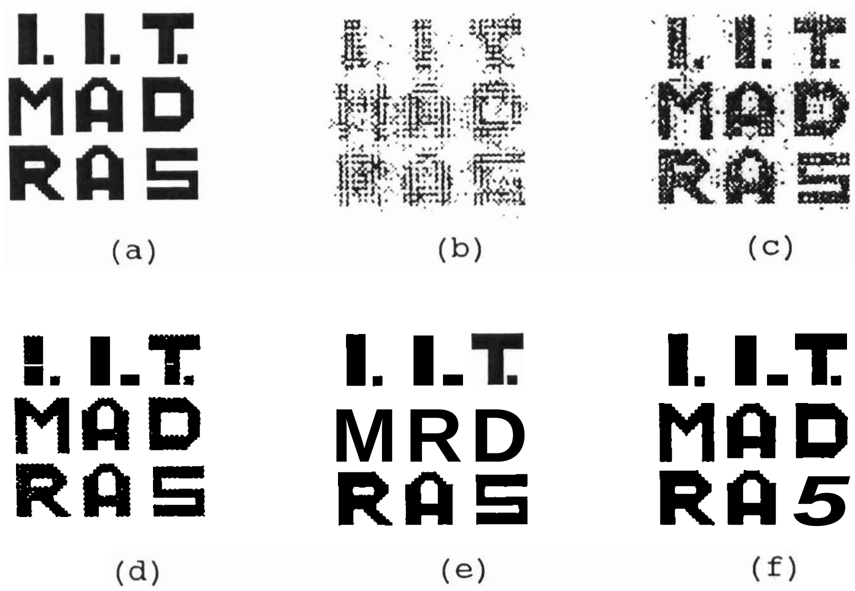



**Fig.4.4** **Images reconstructed from the full phase data collected using 1, 2, 4, 8, and 16 frequencies by a receiver array of 64x64 hydrophones.**
**Number of iterations: 50.**
**(a) Original object 128x128 points.**
**(b) Single frequency.**
**(c) Two frequencies.**
**(d) Four frequencies.**
**(e) Eight frequencies.**
**(f) Sixteen frequencies.**

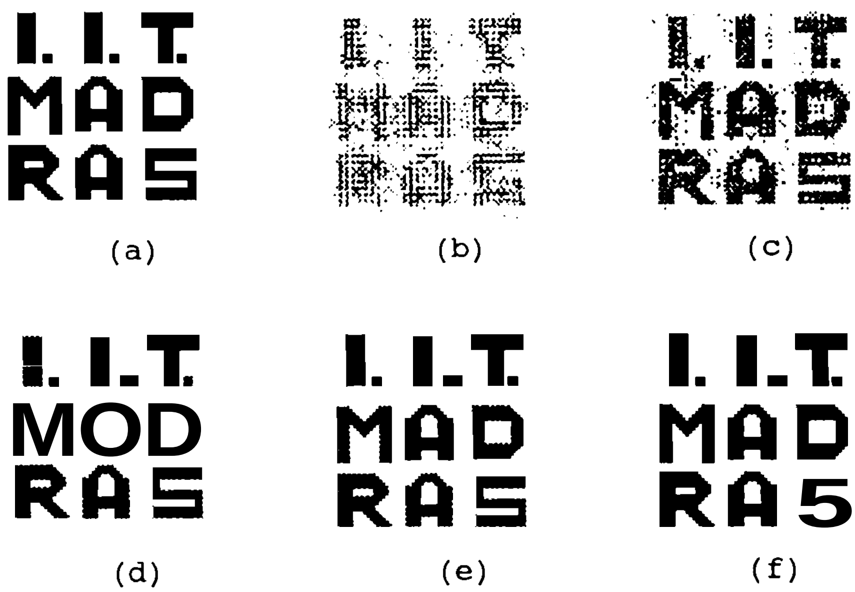



**Fig.4.4** **Images reconstructed from the full phase data collected using 1, 2, 4, 8, and 16 frequencies by a receiver array of 64x64 hydrophones.**
**Number of iterations: 50.**
**(a) Original object 128x128 points.**
**(b) Single frequency.**
**(c) Two frequencies.**
**(d) Four frequencies.**
**(e) Eight frequencies.**
**(f) Sixteen frequencies.**

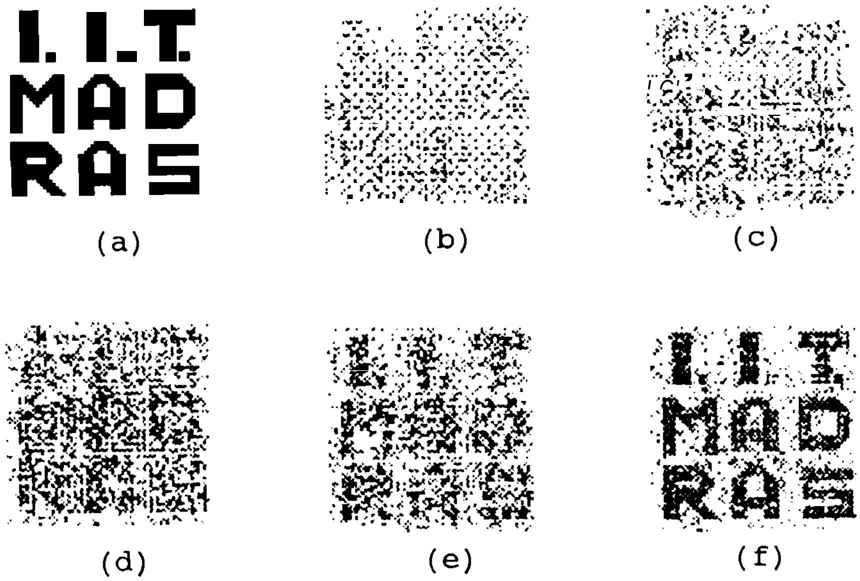



**Fig.4.5** Images reconstructed from the full phase data collected using **1, 2,** 4, **8,** and **16** frequencies by a receiver array of **32x32** hydrophones.
Number of iterations: 50.
(a) Original object **128x128** points.
(b) Single frequency.
(c) Two frequencies.
(d) Four frequencies.
(e) Eight frequencies.
(f) Sixteen frequencies.

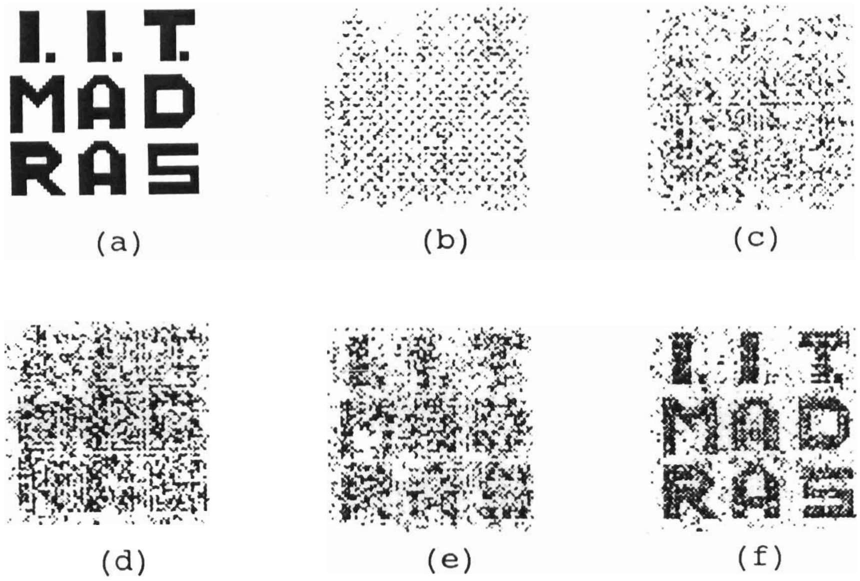



Fig.4.5 Images reconstructed from the full phase data collected using 1, 2, 4, 8, and 16 frequencies by a receiver array of 32x32 hydrophones.
Number of iterations: 50.
(a) Original object 128x128 points.
(b) Single frequency.
(c) Two frequencies.
(d) Four frequencies.
(e) Eight frequencies.
(f) Sixteen frequencies.

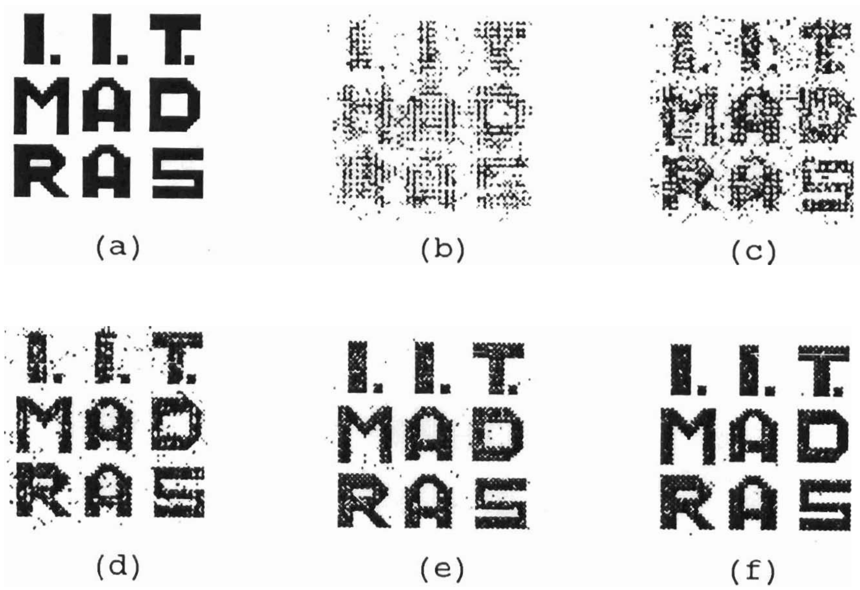


**Fig.4.6** Images reconstructed from the 2-bit phase data collected using 1, 2, 4, 8, and 16 frequencies by a receiver array of 64x64 hydrophones.
Number of iterations: 50.
(a) Original object 128x128 points.
(b) Single frequency.
(c) Two frequencies.
(d) Four frequencies.
(e) Eight frequencies.
(f) Sixteen frequencies.

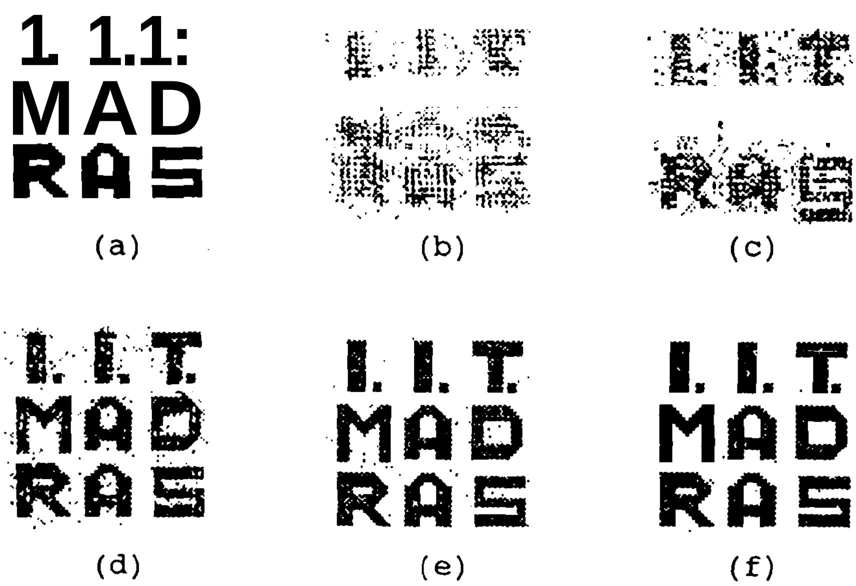


Fig.4.6 Images reconstructed from the 2-bit phase data collected using 1, 2, 4, 8, and 16 frequencies by a receiver array of 64x64 hydrophones.
Number of iterations: 50.
(a) Original object 128x128 points.
(b) Single frequency.
(c) Two frequencies.
(d) Four frequencies.
(e) Eight frequencies.
(f) Sixteen frequencies.

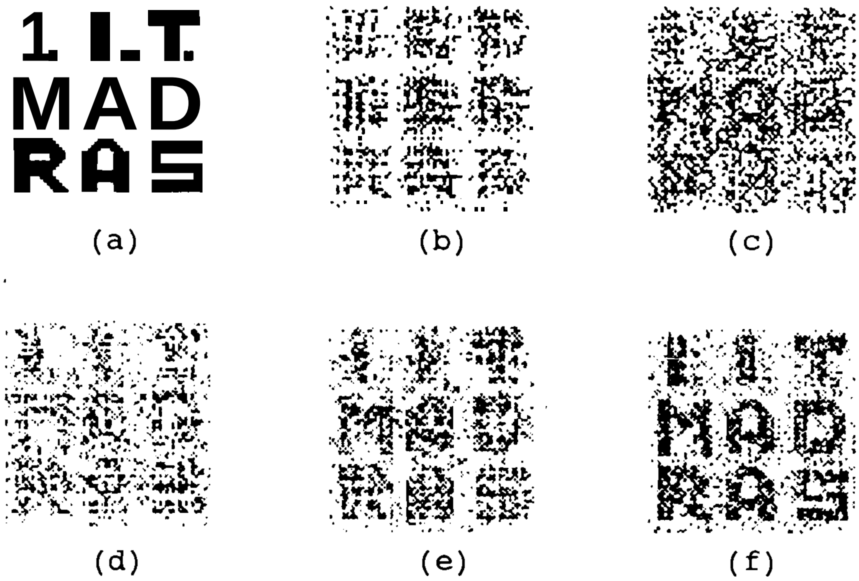


Fig.4.7 Images reconstructed from the 2-bit phase data collected using 1, 2, 4, 8, and 16 frequencies by a receiver array of 32x32 hydrophones.
Number of iterations: 50.
(a) Original object 128x128 points.
(b) Single frequency.
(c) Two frequencies.
(d) Four frequencies.
(e) Eight frequencies.
(f) Sixteen frequencies.

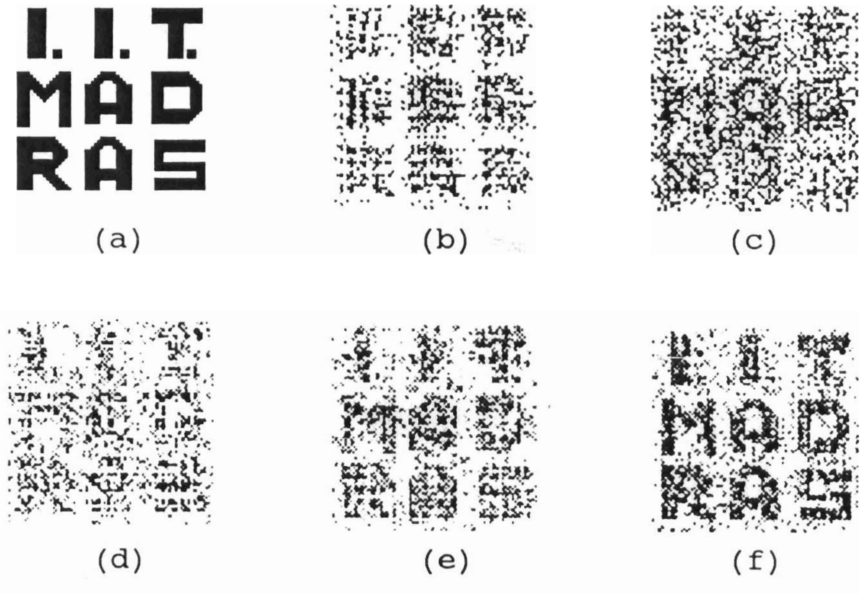
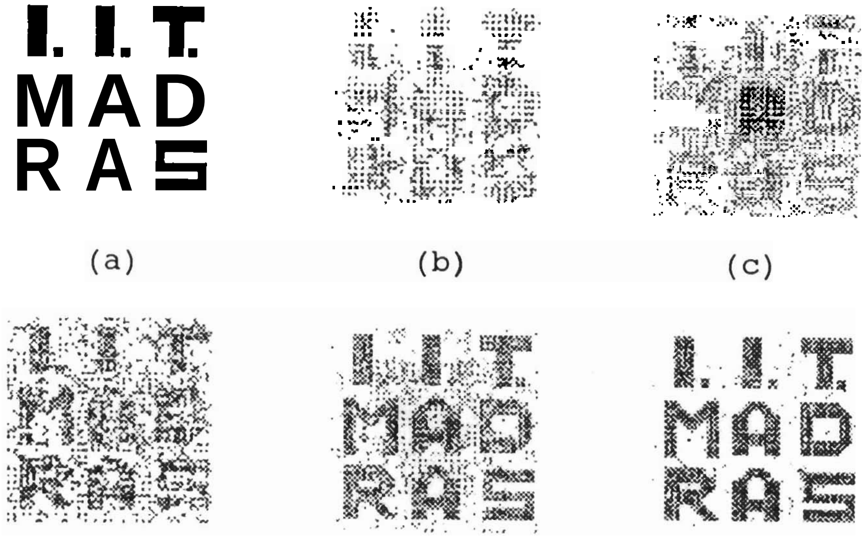



**Fig.4.7** Images reconstructed from the 2-bit phase data collected using **1, 2,** 4, **8,** and **16** frequencies by a receiver array of **32x32** hydrophones.
Number of iterations: 50.
(a) Original object **128x128** points.
(b) Single frequency.
(c) Two frequencies.
(d) Four frequencies.
(e) Eight frequencies.
(f) Sixteen frequencies.



(a) (b) (c)

(d) (e) (f)

**Fig.4.8** **Images reconstructed from the 1-bit phase data collected using 1, 2, 4, 8, and 16 frequencies by a receiver array of 64x64 hydrophones.**
**Number of iterations: 50.**
**(a) Original object 128x128 points.**
**(b) Single frequency.**
**(c) Two frequencies.**
**(d) Four frequencies.**
**(e) Eight frequencies.**
**(f) Sixteen frequencies.**

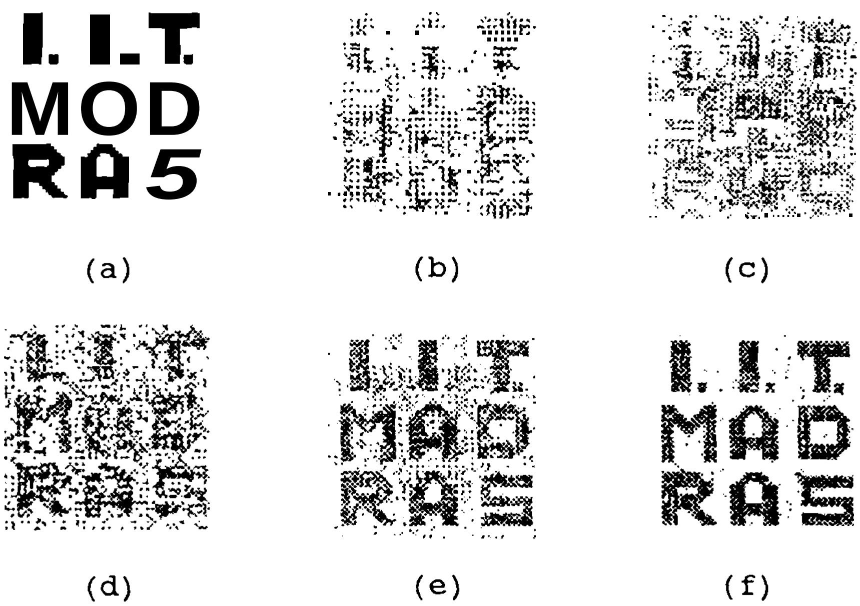



**Fig.4.8** **Images reconstructed from the 1-bit phase data collected using 1, 2, 4, 8, and 16 frequencies by a receiver array of 64x64 hydrophones.**
**Number of iterations: 50.**
**(a) Original object 128x128 points.**
**(b) Single frequency.**
**(c) Two frequencies.**
**(d) Four frequencies.**
**(e) Eight frequencies.**
**(f) Sixteen frequencies.**

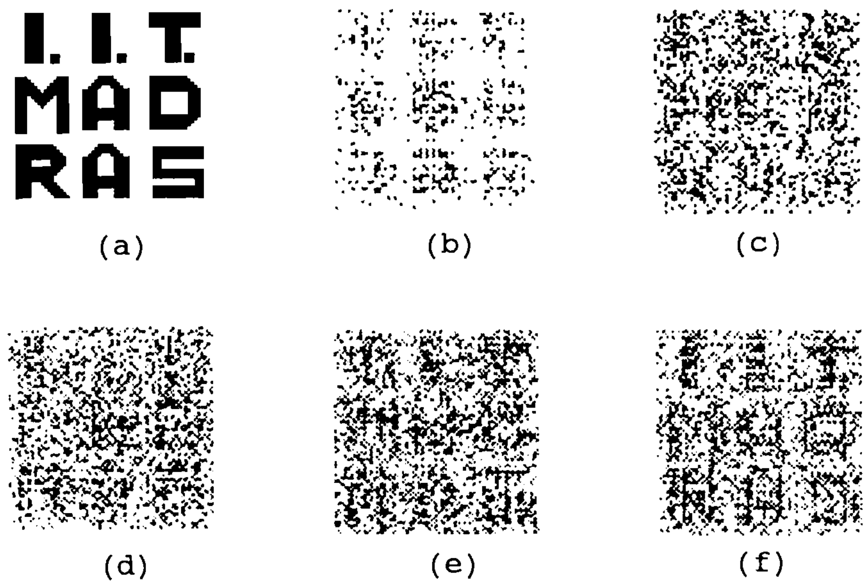



**Fig.4.9** Images reconstructed from the 1-bit phase data collected using 1, 2, 4, 8, and 16 frequencies by a receiver array of 32x32 hydrophones.
Number of iterations: 50.
(a) Original object 128x128 points.
(b) Single frequency.
(c) Two frequencies.
(d) Four frequencies.
(e) Eight frequencies.
(f) Sixteen frequencies.

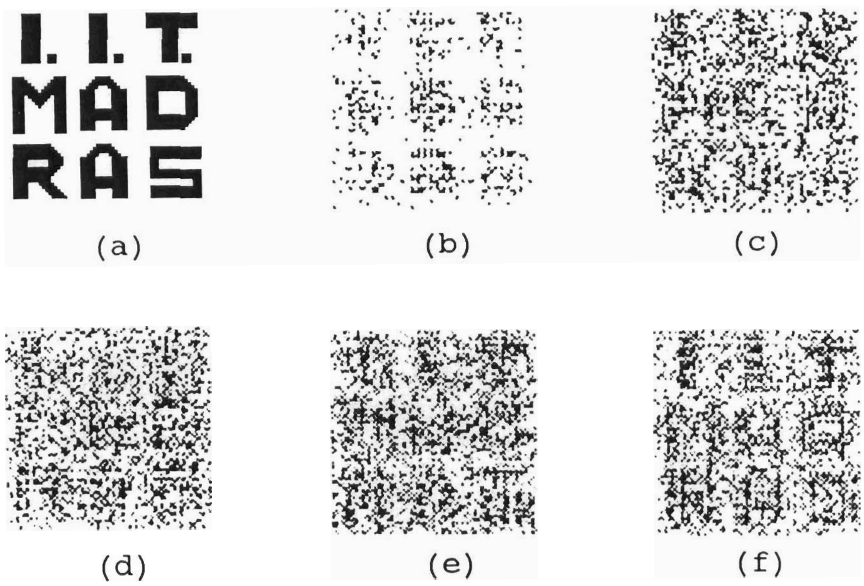


Fig.4.9 Images reconstructed from the 1-bit phase data collected using 1, 2, 4, 8, and 16 frequencies by a receiver array of 32x32 hydrophones.
Number of iterations: 50.
(a) Original object 128x128 points.
(b) Single frequency.
(c) Two frequencies.
(d) Four frequencies.
(e) Eight frequencies.
(f) Sixteen frequencies.

required to produce good results.

## 4.4 SUMMARY

Signal reconstruction from the quantized phase data reduces the measurement complexity at the receiver end. But this technique requires a large number of data points for good reconstruction. Usually the receiver arrays contain small number of receiver elements. Quantized phase data collected from such receiver arrays is not sufficient to form high resolution images. Usually synthetic aperture techniques are used for signal interpolation to improve the resolution of the image. We have used a recently proposed method for signal interpolation for image formation from quantized phase data. This method uses the data collected at various frequencies to form convex sets. Then a suitable POCS algorithm is used for signal reconstruction from these sets. The data collected at multiple frequencies helps to decrease the size of the intersection of the convex sets formed from the available information. Therefore the solution formed from this data is closer to the original signal. A similar algorithm can be applied for signal reconstruction from the quantized phase data measured at a small receiver array using multiple frequencies. This technique reduces the measurement as well as the size complexity of the receiver array. We shall see in the next chapter that the effects of noise in the received data are also reduced by using the phase quantization schemes for image reconstruction.

# CHAPTER 5

# NOISE REDUCTION USING QUANTIZED PHASE INFORMATION

## 5.1 NOISE REDUCTION IN IMAGE FORMATION PROBLEMS

In most array processing problems, the data to be processed to form an image is noisy. If the signal is digital, and the additive noise magnitude is small as compared to the magnitude differences between the various digitization levels, the effect of noise can be reduced easily. For example, consider a binary signal and assume that the two levels of digitization are represented by amplitudes of 1 and 2 units respectively. Then noise upto a maximum magnitude level of 0.5 units is tolerable as a simple thresholding scheme can be used to eliminate the noise completely. But when the noise magnitude is large, the signal can be corrupted. In case of analog signals, any amount of noise corrupts the signal values. This effect is not easily reduced even when the noise magnitude is small. In acoustic imaging where image resolution is usually poor, the addition of noise makes the object recognition more difficult. Therefore study of the effects of noise assumes great significance. In the course of this work, image reconstruction refers to the reproduction of certain object features like edge information, regions of uniform gray-levels, etc. These features are important as they are sufficient for a human observer to recognize the object unambiguously, in most cases. Noise will mean an unwanted additive signal that makes it difficult to recognize the object features mentioned above.

The problem of noise in signals has been studied extensively [44]-[46]. We can model the problem of noise in the following way. Let $\mathbf{g}_o$ be the original signal. The propagation of the signal over the channel can be modelled as a transformation of the original signal. Using an array of hydrophones, this transformed data is measured at the receiver end. The received data $\mathbf{g}$ can, therefore, be written as

$$\mathbf{g} = \mathbf{H}\mathbf{g}_o \tag{5.1}$$

where $\mathbf{H}$ represents the transformation operator. If $\mathbf{H}^{-1}$ is computable, then $\mathbf{g}_o$ can be recovered from $\mathbf{g}$ by using

$$\mathbf{g}_o = \mathbf{H}^{-1}\mathbf{g} \tag{5.2}$$

Equation 5.1 holds only in ideal situations. In the presence of noise, the received data can be written as

$$\mathbf{g} = \mathbf{H}\mathbf{g}_o + \mathbf{n} \tag{5.3}$$

where $\mathbf{n}$ may be the channel and circuit noise. Now the recovery of the original signal depends on the knowledge of n. $\mathbf{n}$ is a random process, therefore one cannot determine its value at each sample. We can at most have an estimate of the noise statistics. But we cannot simulate noise using those statistics and subtract it from the measured data, as this may increase the noise effects. Therefore, some other techniques are required for noise cleaning.

A number of image processing techniques have been developed for noise cleaning in the image domain. A variety of filters have been designed for this purpose [47],[48]. These algorithms do the processing on the image formed after the signal processing stage. If the resolution of the image reconstructed at the output of the the signal processing stage is poor, then the noise reduction techniques may not be applicable.

Keeping this in mind, we have tried to investigate whether noise reduction is possible during the signal processing stage itself. In Chapter 3 we explained that it is possible to recover the original information from the quantized phase information of the received data. An iterative algorithm based on the POCS technique was also given for the signal reconstruction from this quantized phase information. In this chapter we show how the quantized phase information helps to reduce the effects of noise in the reconstructed image.

## 5.2 RECONSTRUCTION FROM QUANTIZED NOISY PHASE DATA

In this section we show that the phase quantization schemes proposed in Chapter 3 can be used for noise reduction in acoustic imaging. We had stated that 2-bit phase information is equivalent to quantizing the phase of the received data to four levels. The quantization scheme is given in section 3.2. The phase is quantized to $\pi/4$, $3\pi/4$, $5\pi/4$, and $7\pi/4$, when the complex vectors fall in the first, second, third or the fourth quadrants, respectively.

Theorem 3.3 states the conditions under which a signal can be recovered from the 2-bit phase of the received data. Let the noise-free received data be represented by $u(r,s)$.

Let the noise signal be represented by a two-dimensional real array $n(r,s)$. Then the noisy received data values are given by:

$$v(r,s) = u(r,s) + n(r,s) \tag{5.4}$$

The noisy siqnal $v$ differs from the noise-free signal $u$ in magnitude as well as phase. This can be seen by taking a sample point, say $u(r_i,s_j)$, in the received data. Corresponding noise value is $n(r_i,s_j)$ ($n_{ij}$ for convenience).

Let $u(r_i,s_j) = a \cos\theta + j\, b \sin\theta$. Then the received data value will be

$$v(r_i,s_j) = (a \cos\theta + n_{ij}) + j\, b \sin\theta \tag{5.5}$$

The magnitude of $v(r_i,s_j)$ is $((a + n_{ij})^2 + b^2)^{1/2}$ and its phase value is $\tan^{-1}(b \sin\theta \,/\, a \cos\theta + n_{ij})$. Therefore at the point $(r_i,s_i)$, the noisy signal value differs from the original signal value in both phase and magnitude. Fig.5.1 shows graphically that due to addition of a real quantity, both the magnitude and phase components of the original complex vector a can change. Therefore if the received data is noisy, and only the phase is measured at the receiver end, the measured values will be different from the actual values.

In Chapter 2 we have given an iterative algorithm for signal recovery from only the phase of the received data for acoustic imaging applications. During each iteration, the phase of the simulated data is corrected by taking the projection onto the convex set consisting of all signals with a given set of phase values. This convex set is formed from the phase of the received data. If the phase of the received

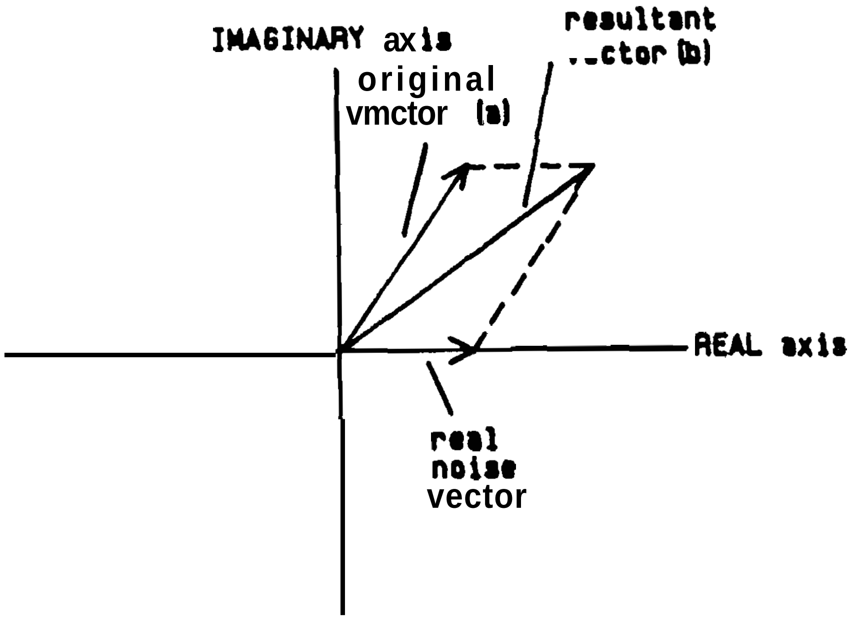



**Fig.5.1** This figure shows that both the phase and the magnitude of a complex vector can change due to the addition of real noise.

data is incorrect, due to the presence of noise, the original signal will not be a member of this set. The intersection of the convex sets formed from the phase of the received data and from the finite support constraint will also not contain the original signal. Therefore the iterative algorithm cannot converge to the original signal. This point is illustrated in Fig.5.2. It shows the two convex sets and that the original signal lies outside the intersection of the two. The algorithm converges to a point in the intersection set but not to the actual solution.

We will now show that a two-dimensional real signal can be recovered from only the 1-bit Fourier transform phase information even in the presence of noise, if the noise level is not very high. Then we will show that a similar argument can be applied for image reconstruction from noisy received data.

We have seen earlier that the addition of noise alters both the phase and magnitude of the received data samples. Let us assume that the noise signal is of low energy. Then it is likely that most of the Fourier transform samples will have the same 1-bit phase as the noise-free data would have had. This is illustrated in Fig.5.3. One of the data samples with value $a\exp(j\theta)$ is changed to $b\exp(j\phi)$ (for some constants a and b) due to the addition of a real noise component n. But we see that both $a\exp(j\theta)$ and $b\exp(j\phi)$ lie in the same half-plane and therefore have the same 1-bit phase as the original signal. The half-planes of interest are those demarcated by the imaginary axis.

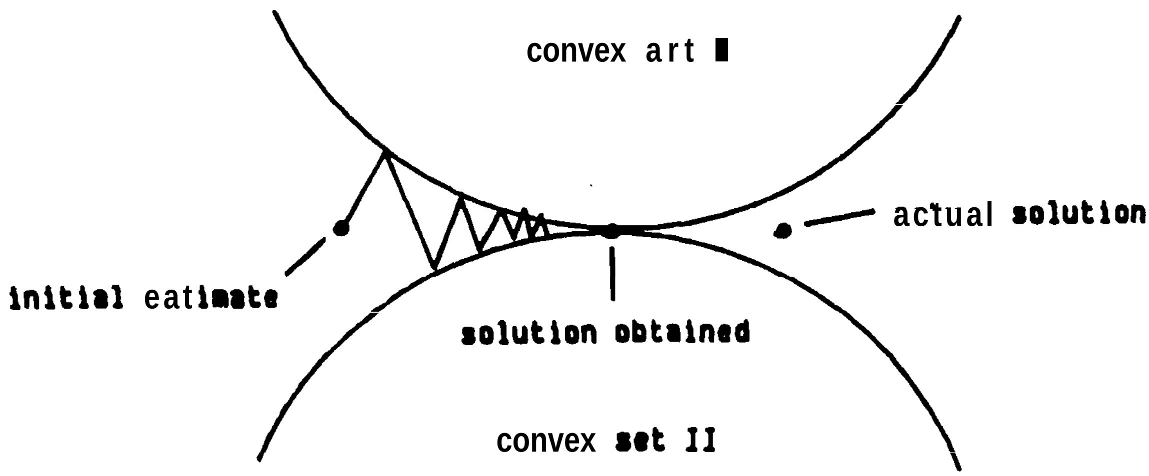



Fig.5.2 This figure shows two complex sets - set I and set II. The desired solution point does not lie in the intersection of the two sets. Therefore, irrespective of the initial estimate chosen, the POCS algorithm cannot converge to the required solution. Such situations can arise due to the addition of noise in the data.

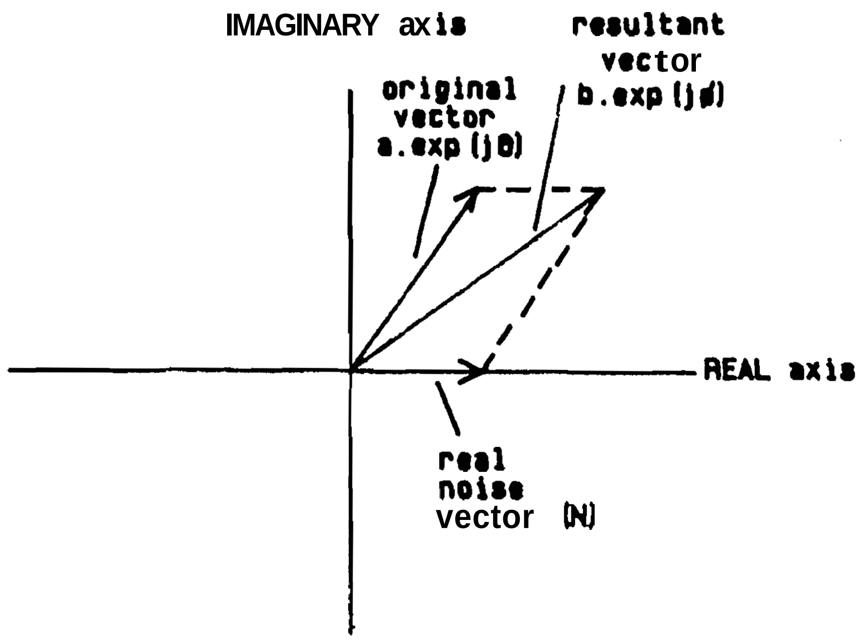



Fig.5.3 This figure shows the case when a complex vector a.exp(jθ) is changed to another vector b.exp(jø) due to the addition of a real noise vector (n). But both the original and the resultant vectors lie in the same half-plane, and therefore, have the same 1-bit phase value. If the additive noise vectors have small magnitudes, the 1-bit phase values of most of the data samples may be unaffected.

This concept can be explained numerically also. Let the phase at each sample be represented as a binary number. Quantizing the phase of the data to two levels is equivalent to retaining the most significant bit (MSB) of the phase value. Assume that a particular sample has a phase given by 10010111 (this represents an 8-bit precision in phase measurement). In this case the quantized phase value is 1 (MSB of the binary number). Let the addition of noise add a small amount of phase to the original value. Say the additive phase is 00001100. Then the resultant phase value at that sample becomes 10100011. The quantized phase (MSB) of the resultant value is still 1. The addition of a small amount of noise has not caused any change in the quantized phase value for that sample. Thus the 1-bit phase information for that particular sample is the same with or without the presence of noise. If the noise level is less, most of the samples may not be affected by it when their phase is quantized. But whenever the magnitude of a sample is small or when the complex vector lies very near the imaginary axis, a small amount of noise can change the quantized phase information to the wrong value.

A complex-valued signal can be recovered from the zero-crossing information of the real and imaginary parts of its Fourier transform. If additive noise is real, only real part of the data values are affected. The odd component of the two-dimensional complex signal is recovered from the imaginary part of the received data. Since the real additive noise does not affect the imaginary part of the received data, the odd component of the signal can be recovered as it would have been in the absence of noise. The even component of the signal is recovered from zero-crossing points of the real part of the received data. Noise affects the real part of the data. The points where the signal went to zero may not be the zero-crossing points after addition of noise. Therefore there will be some uncertainty in the determination of the zero-crossing points. This implies that the iterative algorithm may not converge to the original signal. But as we have seen, the addition of small noise values may not have much effect on the quantized phase values of the sampled signal. Therefore, during each iteration, the correct sign will be substituted at most of the points. This may be compared with the reconstruction using full phase information. The measured phase values are incorrect at all the points where the noise is non-zero. During each iteration, these incorrect values are used for phase correction. It is evident that the error in the measured data is propagated further. Therefore we can expect the images obtained from the quantized phase information to be better than those obtained from the full phase information. The experimental results in the next section will show that in fact the images obtained from the quantized phase of the noisy data reproduce the essential features of the original signal, whereas those obtained from full phase data do not. Since images are recognized from their features, a technique that reproduces these is acceptable even though it does not guarantee exact convergence to the original.

The main idea in using the quantized phase information for image formation in the presence of noise is to use a small amount of right information rather than a large amount of incorrect information. Even if the full phase measurement is available, it is better to use quantized phase information during iterations. The full phase information of the data can be used to form an initial estimate of the signal.

In the case of quantized phase information, we can use some heuristics to improve the performance. A quantized phase value of $\pi$ surrounded by the quantized phase values of $0$ can be substituted by $0$, assuming that it is an error point. Similarly a quantized phase value of $0$ surrounded by the quantized phase values of $\pi$ can be substituted by $\pi$, assuming that it is also an error point. Similar heuristic approach may not be possible when the full phase information is used.

The phase quantization technique can be used with another variation. We state the following result adopted from [34] with $\alpha=0$.

Theorem 5.1 Let $x(n_1,n_2)$ and $y(n_1,n_2)$ be real two-dimensional sequences with support over a finite nonsymmetric half-plane, with $\mathrm{Sign}\{\mathrm{Re}[X(f_1,f_2)]\} = \mathrm{Sign}\{\mathrm{Re}[Y(f_1,f_2)]\}$ for any $\alpha$ such that $\mathrm{Re}\{X(f_1,f_2)-\alpha\}$ takes on both positive and negative values. Also let

$$x(n_1,n_2) = \frac{x(n_1,n_2) + x^*(-n_1,-n_2)}{2} - \alpha\,\delta(n_1,n_2)$$

$$y(n_1,n_2) = \frac{y(n_1,n_2) + y^*(-n_1,-n_2)}{2} - \alpha\,\delta(n_1,n_2)$$

$$\text{where } \delta(n_1,n_2) = \begin{cases} 1 & \text{if } (n_1,n_2) = (0,0) \\ 0 & \text{otherwise} \end{cases}$$

If $\hat{X}(z_1,z_2)$ and $\hat{Y}(z_1,z_2)$ are nonfactorable, then $\hat{x}(n_1,n_2) = c\, \hat{y}(n_1,n_2)$ for $(n_1,n_2) \neq 0$, and $\hat{x}[0,0]-\alpha = c\,[\hat{y}(0,0)-\alpha]$ for some positive constant c.

This theorem states that if the phase quantization is done after subtracting a constant $\alpha$ from all the samples, the signal can be recovered from this quantized phase information. In effect, the crossing of an arbitrary threshold $\alpha$ , and not necessarily zero can be considered for phase quantization. This result can be used for phase quantization of noisy signals. If there is a particular value around which the signal rises or falls steeply, it can be used as the threshold for quantization. This way, even when the signal samples are very close to each other, the effect of noise on the samples around that threshold will be smaller as compared to the effect around the actual zero-crossing contour. Therefore, the contour where the signal crosses the threshold $\alpha$ will be known more accurately than the region where the real part of the signal crosses zero. Hence the reconstructed image from the phase quantized in this manner can be expected to be better.

In Chapter 4 it was shown that signal reconstruction is better if data is collected by transmitting waves of several frequencies. In this chapter we have seen that the quantized phase information can be used to reduce the effects of noise in the data on the reconstructed image. We have studied the use of the quantized phase information collected at several frequencies for noise reduction. The experimental results are discussed in the next section.

## 5.3 ILLUSTRATION OF NOISE REDUCTION

In this chapter we have proposed that noise effects can be reduced by using quantized phase of the received data. In this section we give the results of experimental studies on the image reconstruction from the quantized phase of noisy received data.

The object shown in Fig.3.2 was used for the experimental studies. Figs.5.4(a) to 5.4(c) show the images reconstructed from full phase, 2-bit phase, and 1-bit phase information, respectively. These images were obtained after 1 iteration of the POCS algorithm for the data collected at eight frequencies by a receiver array of 64x64 points. This data was noise-free. The corresponding images obtained after 25 iterations are shown in Figs.5.4(d) to 5.4(f). It can be seen that the images formed from full phase information converge faster towards the original as compared to the image obtained from 2-bit phase data. Reconstruction from the 1-bit phase data is shown just for comparison. In general we do not expect to get good images from noisy 1-bit phase data.

Fig.5.5 shows the images obtained when the data was corrupted with normally distributed noise with SNR of 12db. Figs.5.5(a) to 5.5(c) show the images formed from full phase, 2-bit phase, and 1-bit phase data, respectively, after 1 iteration of the POCS algorithm. The images obtained after 25 iterations are shown in Figs.5.5(d) to 5.5(f). We see from these figures that the images formed from 2-bit phase information are as good as those formed from full phase information.

The use of quantized phase information is more evident for higher noise levels. Fig.5.6 illustrates this point. In this case the **SNR** was -2db. The order of images in Figs.5.6(a) to 5.6(f) is as in Figs.5.4 and 5.5. We see that the image formed from 2-bit phase data shows convergence towards the original. The image formed from full phase information does not show any such convergence. Fig.5.7 shows the corresponding images reconstructed with **SNR** = -10db.

The noise levels for images in Figs.5.6 and 5.7 were higher than that for the images in Fig.5.5. The 2-bit phase data performs better than the full phase data only for moderate noise levels. When the noise in the data is very high, reconstruction is not possible from both the full and 2-bit phase data. Figs.5.8(a) to 5.8(c) show the images obtained from full phase data, 2-bit phase, and 1-bit phase data, respectively, when the **SNR** was -30db. These images were obtained after 1 iteration of the **POCS** algorithm. The corresponding images obtained after 25 iterations are given in Figs.5.8(d) to 5.8(f). We see that at high noise levels, it is not possible to reconstruct the image from both the full phase and 2-bit phase data.

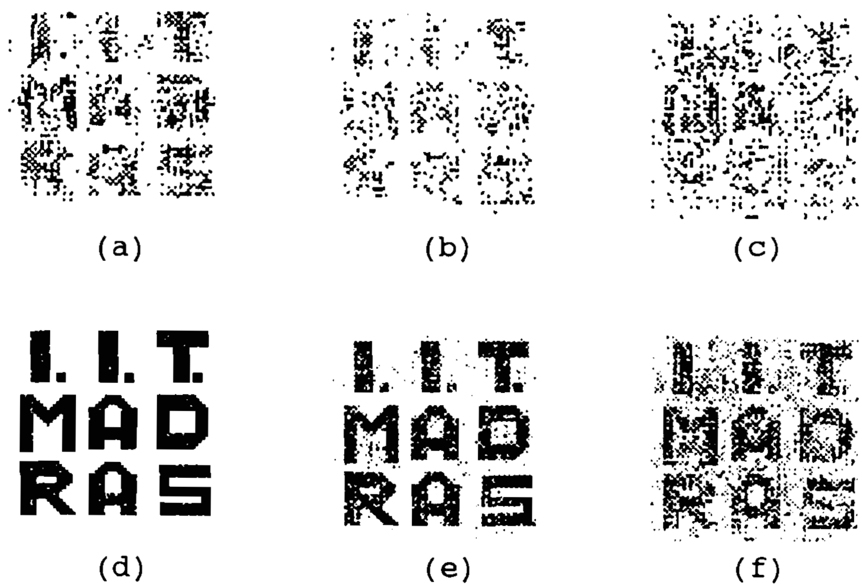



**Fig.5.4** The images reconstructed from full phase, 2-bit phase, and 1-bit phase of noise-free data.
Original object 128x128 points.
Number of receiver elements 64x64.
Number of frequencies used: 8.
(a) From full phase after 1 iteration.
(b) From 2-bit phase after 1 iteration.
(c) From 1-bit phase after 1 iteration.
(d) From full phase after 25 iterations.
(e) From 2-bit phase after 25 iterations.
(f) From 1-bit phase after 25 iterations.

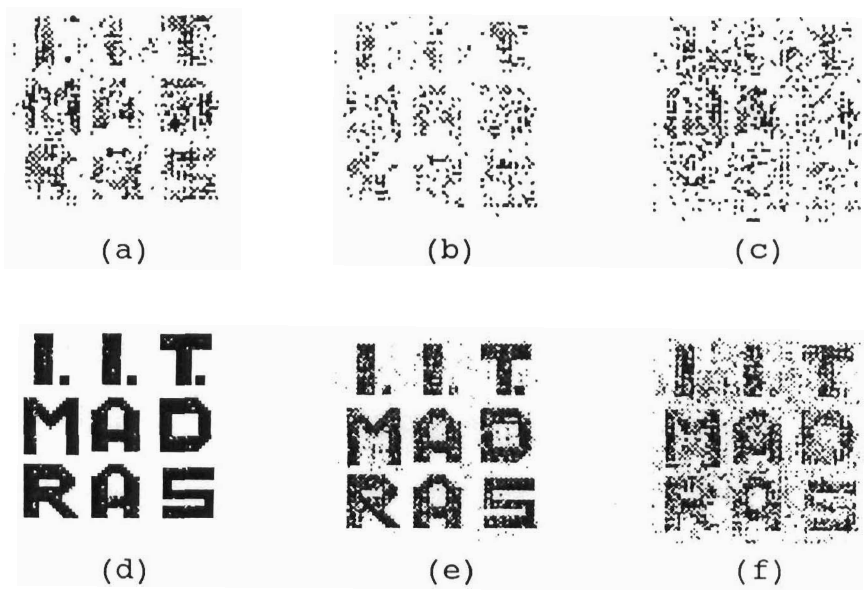



**Fig.5.4** The images reconstructed from full phase, 2-bit phase, and 1-bit phase of noise-free data.
Original object **128x128** points.
Number of receiver elements 64x64.
Number of frequencies used: 8.
(a) From full phase after 1 iteration.
(b) From 2-bit phase after 1 iteration.
(c) From 1-bit phase after 1 iteration.
(d) From full phase after 25 iterations.
(e) From 2-bit phase after 25 iterations.
(f) From 1-bit phase after 25 iterations.

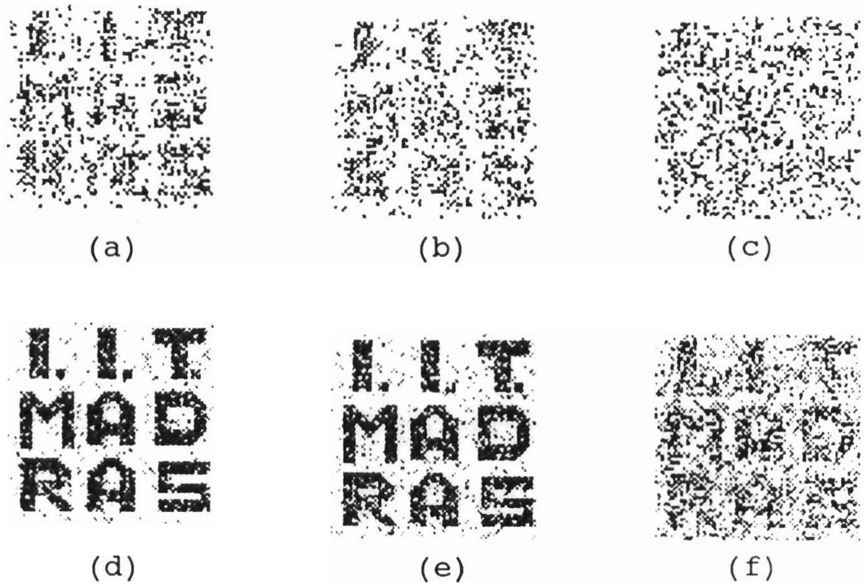



**Fig.5.5** The images reconstructed from full phase, 2-bit phase, and 1-bit phase data with SNR = 12db.
Original object 128x128 points.
Number of receiver elements 64x64.
Number of frequencies used: 8.
(a) From full phase after 1 iteration.
(b) From 2-bit phase after 1 iteration.
(c) From 1-bit phase after 1 iteration.
(d) From full phase after 25 iterations.
(e) From 2-bit phase after 25 iterations.
(f) From 1-bit phase after 25 iterations.

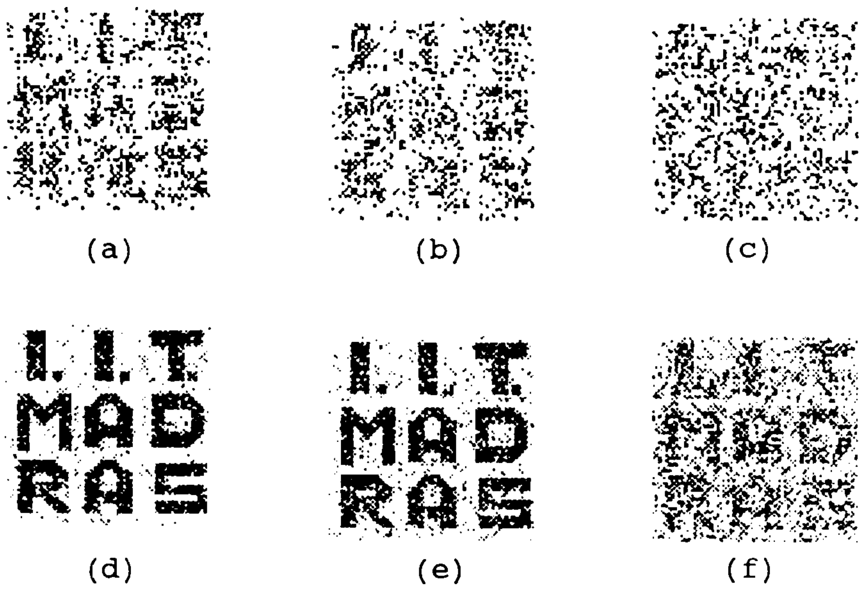


Fig.5.5 The images reconstructed from full phase, 2-bit phase, and 1-bit phase data with SNR = 12db.
Original object 128x128 points.
Number of receiver elements 64x64.
Number of frequencies used: 8.
(a) From full phase after 1 iteration.
(b) From 2-bit phase after 1 iteration.
(c) From 1-bit phase after 1 iteration.
(d) From full phase after 25 iterations.
(e) From 2-bit phase after 25 iterations.
(f) From 1-bit phase after 25 iterations.

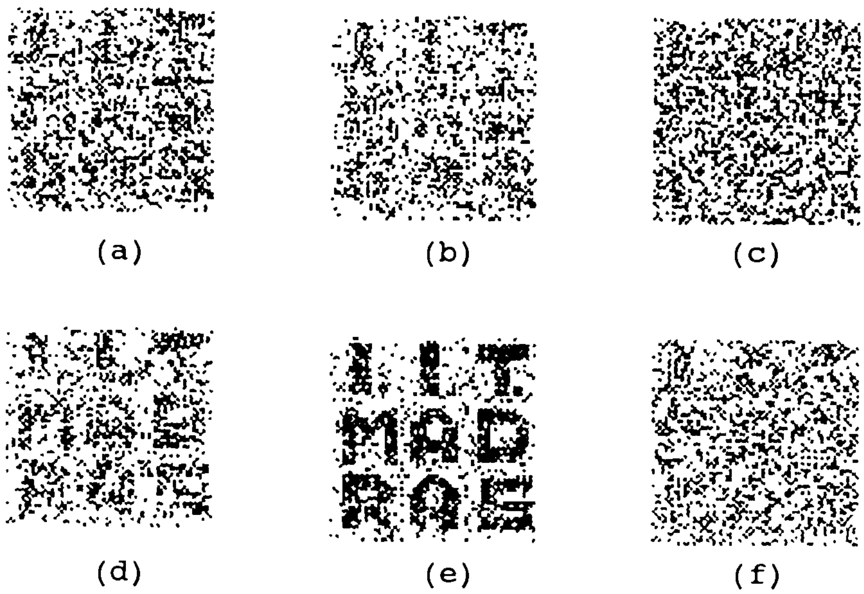


**Fig.5.6** The images reconstructed from full phase, 2-bit phase, and 1-bit phase data with SNR = -2db.
Original object 128x128 points.
Number of receiver elements 64x64.
Number of frequencies used: 8.
(a) From full phase after 1 iteration.
(b) From 2-bit phase after 1 iteration.
(c) From 1-bit phase after 1 iteration.
(d) From full phase after 25 iterations.
(e) From 2-bit phase after 25 iterations.
(f) From 1-bit phase after 25 iterations.

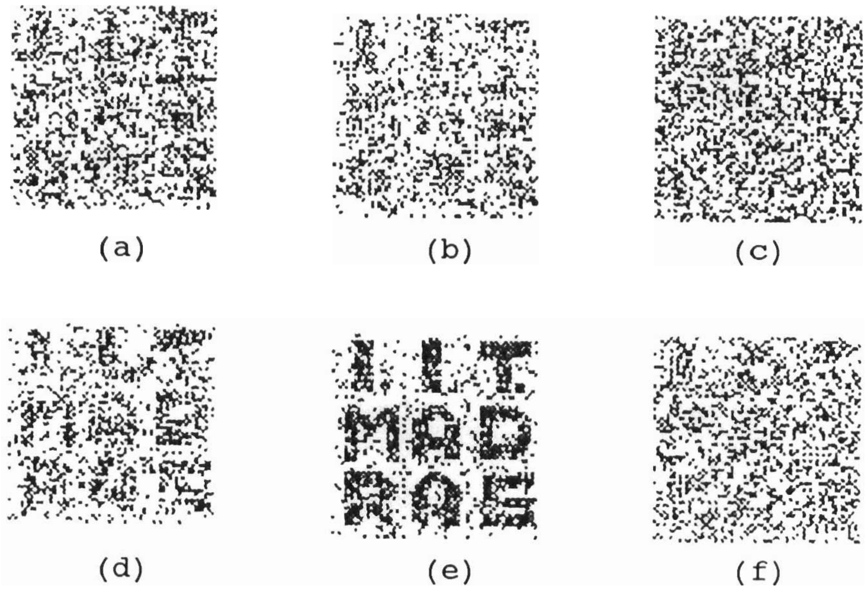


Fig.5.6 The images reconstructed from full phase, 2-bit phase, and 1-bit phase data with SNR = -2db.
Original object 128x128 points.
Number of receiver elements **64x64.**
Number of frequencies used: 8.
(a) From full phase after 1 iteration.
(b) From 2-bit phase after 1 iteration.
(c) From 1-bit phase after 1 iteration.
(d) From full phase after 25 iterations.
(e) From 2-bit phase after 25 iterations.
(f) From 1-bit phase after 25 iterations.

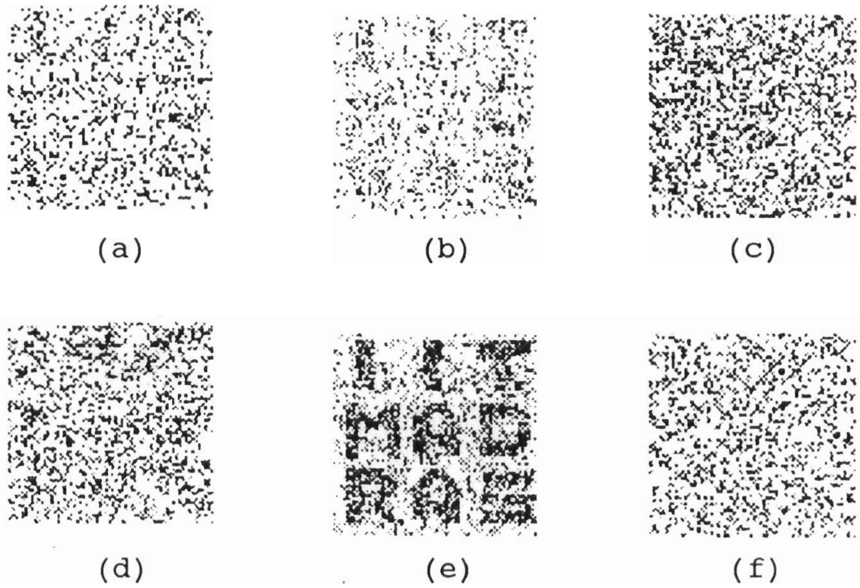



Fig. 5.7 The images reconstructed from full phase, 2-bit phase, and 1-bit phase data with SNR = -10db.
Original object 128x128 points.
Number of receiver elements 64x64.
Number of frequencies used: 8.
(a) From full phase after 1 iteration.
(b) From 2-bit phase after 1 iteration.
(c) From 1-bit phase after 1 iteration.
(d) From full phase after 25 iterations.
(e) From 2-bit phase after 25 iterations.
(f) From 1-bit phase after 25 iterations.

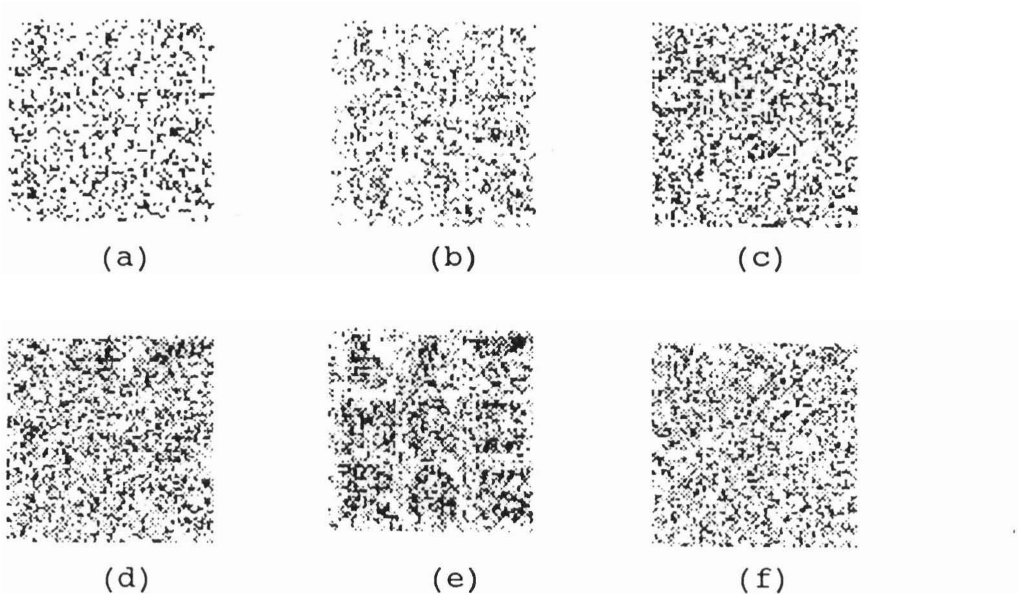


Fig.5.8 The images reconstructed from full phase, 2-bit phase, and 1-bit phase data with SNR = -30db.
Original object 128x128 points.
Number of receiver elements 64x64.
Number of frequencies used: 8.
(a) From full phase after 1 iteration.
(b) From 2-bit phase after 1 iteration.
(c) From 1-bit phase after 1 iteration.
(d) From full phase after 25 iterations.
(e) From 2-bit phase after 25 iterations.
(f) From 1-bit phase after 25 iterations.

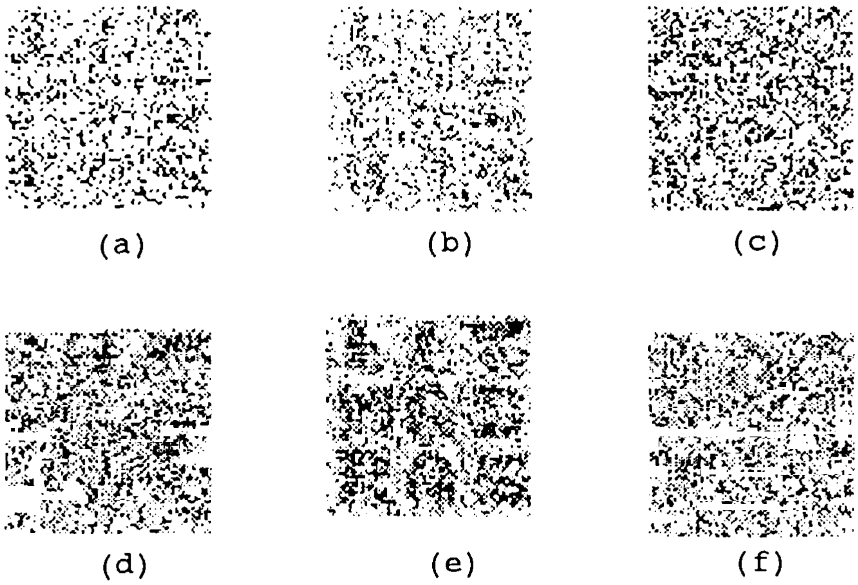


**Fig.5.8** The images reconstructed from full phase, 2-bit phase, and 1-bit phase data with **SNR** = -30db.
Original object 128x128 points.
Number of receiver elements 64x64.
Number of frequencies used: 8.
(a) From full phase after 1 iteration.
(b) From 2-bit phase after 1 iteration.
(c) From 1-bit phase after 1 iteration.
(d) From full phase after 25 iterations.
(e) From 2-bit phase after 25 iterations.
(f) From 1-bit phase after 25 iterations.

## 5.4 SUMMARY

The technique of signal reconstruction from quantized phase information can be used for noise reduction. Since the signal is complex valued, the signs of both the real and imaginary parts of the received data are required for reconstruction. Real additive noise affects only the real part of the received data. Therefore the odd component of the original signal can be recovered from the imaginary part of the received data as it would have been in the absence of noise. Presence of noise causes more uncertainty in the determination of the zero-crossing points of the real part of received data. This causes some error in determination of the even component of the signal. But if the noise energy is low, the 2-bit phase at most of the data points will not be affected, whereas the actual phase measurement at these points will be incorrect. Though by quantizing the phase we retain lesser amount of information at each point, images reconstructed from this are better than those reconstructed from the full phase information.

## RESULTS AND CONCLUSIONS

In this thesis, we have addressed the problem of signal reconstruction from sensor array data. This problem arises in many practical situations. The data available in such situations is finite and discrete due to limited number of receiver elements on the array used for measurements. The data is usually a set of complex numbers. Sometimes, the phase or the magnitude information may not be available, or these quantities may not be measured accurately. These conditions, and the presence of noise in the received data makes the problem of signal recovery as that of reconstruction from partial data. In this thesis, we have proposed techniques for solving some of these problems. The techniques proposed in this work use the POCS algorithm to recover the original signal from the partial information in various domains.

We have shown that under certain circumstances, we do not require full phase information for signal recovery. ~~signals can be recovered from only the phase of the received data~~. Under certain conditions, it is possible to recover two-dimensional complex-valued signals from the phase data. For signal recovery, we do not require full phase information. Phase data quantized to two levels (1-bit phase) or to four levels (2-bit phase) is sufficient for signal recovery in most of the cases. The possibility of signal recovery from full/quantized phase information is useful because the measurement complexity can be reduced if magnitude information is not required. But more number of data samples are required for reconstruction from the phase data. This implies that the receiver array should contain a larger number of receiver elements. This difficulty can be overcome by using a signal interpolation scheme that uses the data collected at several frequencies in the POCS algorithm. With this method it is possible to reconstruct signals from data collected at arrays with a small number of receiver elements.

Signal reconstruction from the quantized phase data collected at several frequencies also reduces the effects of noise in the received data. Noise changes both the magnitude and the phase of the data samples. By quantizing the phase we discard the lower order bits of the phase values. Therefore the effect of noise is reduced. The main idea is to use a small amount of correct information rather than a large amount of incorrect information.

The techniques proposed in this thesis can be applied for a wide variety of signal reconstruction problems. In this work, we have illustrated their use for a simulated acoustic imaging system. From these studies we see that both the measurement and the size complexity of the receiver array can be reduced by using quantized phase data for image reconstruction. But since these algorithms have been tested only for simulated situations, many practical problems that one might face in real situations have been overlooked. The acoustic field data at the receiver end is normally due to three-dimensional objects. The field measurements are subjected to errors and distortion due to medium effects. It is difficult to predict the effects of these factors on the reconstructed images.

The most interesting result of this study is the trade-off between the computational and the measurement complexity. It is quite possible that solutions for problems of information recovery from partial data may be viewed from this angle. Since computation is easier to realize than physical measurement, it may be possible to recover the desired information from the data collected with a relatively simple setup.

# APPENDIX

## THEORY OF PROJECTIONS ONTO CONVEX SETS

In this appendix we give the basic theory of projections onto convex sets and show how some of the available information forms convex sets. The standard POCS algorithm has been used to implement the results developed in this thesis. Most of the results related to formation of convex sets and the methods to take projections onto these convex sets are similar to the work reported in [26]-[29]. The POCS algorithm works by alternate projections onto the various convex sets. Statements of the theorems related to convergence of this algorithm are also given here for sake of completeness.

Definition 1 [47],[48] :- A normed linear space is a linear space $S$, in which to each vector $x$ there corresponds a real number, denoted by $||x||$ and called the norm of $x$, in such a manner that

(1) $||x|| \geq 0$, and $||x|| = 0 \Leftrightarrow x = 0$

(2) $||x+y|| \leq ||x|| + ||y||$

(3) $||ax|| = |a| \, ||x||$ for a constant $a$

Definition 2:- A complete normed linear space is called a Banach space.

Definition 3:- A Hilbert space is a complex Banach space whose norm arises from an inner product, that is, in which there is defined a complex function $(x,y)$ of vectors $x$ and $y$ with the following properties:

(1) $(ax+by,z) = a(x,z) + b(y,z)$

(2) $(x,y) = (y,x)^*$

(3) $(x,x) = ||x||^2$

Definition 4:- The projection of a point x onto a set R of a normed space E is the point $P_R(x)$ such that

$$||x-P_R(x)|| = \inf_{y \in R} ||x-y|| \tag{A.1}$$

Definition 5:- A set C of a normed space R is called a convex set if

$$ax + (1-a)y \in C \text{ for all } 0 \leq a \leq 1 \text{ and } x,y \in C \tag{A.2}$$

Definition 6:- A sequence $x_n$ in a Hilbert space H is said to converge strongly to a point $q \in H$ if

$$\lim_{n \to \infty} ||q-x_n|| = 0 \tag{A.3}$$

The convergence is said to be weak if

$$\lim_{n \to \infty} (x_n,y) = (q,y) \text{ for all } y \in H. \tag{A.4}$$

Lemma 1:- Let C be any convex set in a Hilbert space H and let Px represent the projection of $x \in H$ onto C. Then

$$\mathrm{Re}(x-Px,y-Px) \leq 0 \text{ for all } y \in C \text{ and } x \notin C.$$

Lemma 2:- Let C be any convex set in a Hilbert space H and let Px represent the projection of $x \in H$ onto C. If $x \notin C$ and $Q \in C$, then

$$||Q-Px||^2 < ||Q-x||^2$$

Lemma 3:- Let $C_1$ and $C_2$ be two convex sets in a Hilbert space H, such that they have Q as the only common point. Let the projection operators onto $C_1$ and $C_2$ be $P_1$ and $P_2$ respectively. If $x \notin C_2$, then

$$||Q-P_1P_2x||^2 < ||Q-x||^2$$

Lemma 4:- Let $C_1$ and $C_2$ be two convex sets in a Hilbert space H, such that they have Q as the only common point. Let the projection operators onto $C_1$ and $C_2$ be $P_1$ and $P_2$ respectively. Then starting at any arbitrary point $x \in H$, we have

$$\lim_{n \to \infty} ||Q-(P_1P_2)^n x|| = 0.$$

Our work is related to processing of signals. The set of all signals forms a Hilbert space. The signals are complex valued, in general. For any two signals $x(n)$ and $y(n)$, we define an inner product as

$$(x,y) = \sum_i x[i].y^*[i]$$

Therefore the norm of a vector (signal) $x(n)$ will be

$$||x||^2 = \sum_i x[i].x^*[i]$$

These definitions and lemmas state the important results related to the convergence of the POCS algorithm. The algorithm is used to take projections onto various convex sets. Now we show that various types of constraints used in our work form convex sets.

Statement 1:- Knowledge of the finite support of a signal forms a convex set.

Proof:- Let A represent the set of points contained in the region of finite support known for the signal. Let B represent the region where the signal values are zero (the region outside the finite support). Let x and y be two signals with A as their region of support. Then the values of both x and y are zero in the region B. Consider the signal $z = ax + (1-a)y$ for $0 \le a \le 1$. In the region B, z is also zero. Therefore z also has A as the region of support. Hence the information of the finite support of a signal forms a convex set.

Statement 2:- Knowledge of the phase of a signal at a given set of points forms a convex set.

Proof:- Let the set of points where the phase values are known be called set A. Let the set of all signals that have the specified phase values at points in A be called set C. If x and y are any two signals in C, then they have the same phase values at the points in set A. Consider a signal $z = ax + (1-a)y$ for $0 \leq a \leq 1$. Then the phase of z will be same as the phase of x or y at the points in the set A. The magnitude values may differ. The magnitude of the samples in z will be equal to a times the magnitude of samples of x plus (1-a) times the magnitude of the corresponding samples in y. This is because $a \exp(j\theta) + b \exp(j\theta) = (a+b) \exp(j\theta)$. Therefore, z is also an element of C. Thus C forms a convex set.

This statement is a general one. If phase values of all the signal samples are known, it also defines a convex set.

Statement 3:- The set of all signals that have specific values at a set of points forms a convex set.

Proof:- Let the set of points where the signal values are known be called set A. Let C denote the set of all signals that have the known values at the points in A. Take any $x, y \in C$. Consider a signal $z = ax + (1-a)y$ for $0 \leq a \leq 1$. Since x and y have equal values at the points in A, z also will have the same values at those points. Therefore the knowledge of a few signal samples defines a convex set.

Statement 4:- The set of all signals that have specific 1-bit phase at a given set of points is convex.

Proof:- Two signal samples are said to have equal 1-bit phase values if they both lie on the same side of the imaginary axis. Let x and y be two such vectors (signal samples). Then if $z = ax + (1-a)y$ for $0 \leq a \leq 1$, it is the vector sum of scaled versions of x and y. If x and y lie on the same side of the imaginary axis, then z will also lie on the same side, and will have the same 1-bit phase as x or y. This is illustrated by Fig.A.l. A signal is formed of a number of such complex vectors. Let x and y be two signals whose 1-bit phase is known at a set of points A. If and $z = ax + (1-a)y$, the 1-bit phase values of the samples in z will be equal to the 1-bit phase values in corresponding samples of x or y at the points in A. Therefore the knowledge of 1-bit phase of a signal defines a convex set.

As a special case of this statement, the knowledge of 1-bit phase at all the points in a signal also forms a convex set. On the same lines we have

Statement 5:- The set of all signals that have specific 2-bit phase at a given set of points is convex.

Now we show how to take projections onto the convex sets defined above. Projections onto the convex sets are used in the POCS algorithm used in this work.

(1) Projection onto the convex set formed from the finite support constraint.

Let a signal be known to be zero outside the region R.

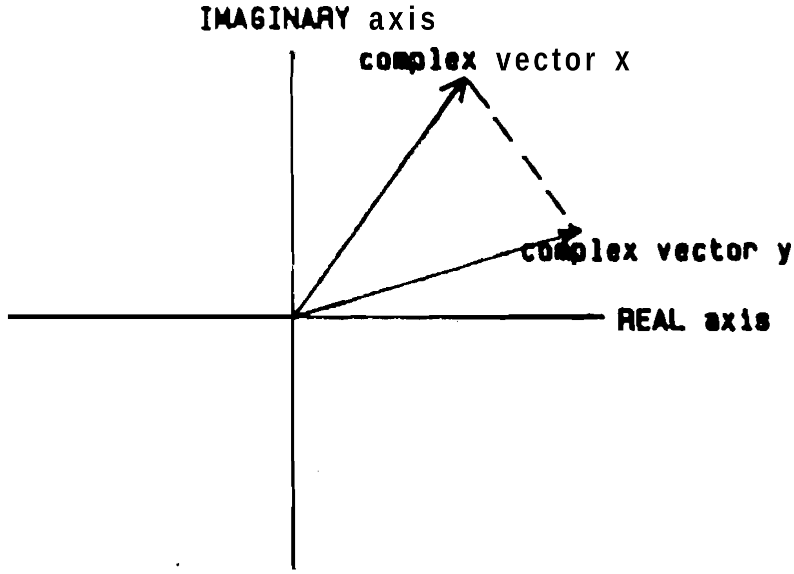



Fig.A.1 The solid lines show two complex vectors x and y that have the same 1-bit phase values. Any other vector $z$ defined as $z = a\,x + (1-a)\mathbf{y}$ for $0 < a < 1$, will lie along the dotted line. Since it is-also in the same half-plane as x or y, it has the same 1-bit phase value as **x** or *y*. Therefore, the knowledge of 1-bit phase defines a convex set.

If $x$ is a signal which is non-zero outside R, then it does not satisfy the finite support constraint. If $c_f$ is the convex set containing all the signals that have R as their finite support, then $x$ is not an element of $c_f$. The projection of $x$ onto R is an element $x_r$ of R such that $|x - x|$ is minimum. For a one-dimensional case, it is equivalent to minimizing $||z[i]||$ where

$$z[i] = x[i]-x_r[i] \tag{A.5}$$

It is known that $x_r$ has zero values outside the region R. To minimize the expression we have the freedom to choose the values of $x_r$ in the region R. It can be verified that the expression is minimized if $x_r[i] = x[i]$ in the region of finite support. Therefore, to take the projection of a signal $x$ onto a convex set defined by the knowledge of a finite support R, we must set the values outside R to zero and retain all other values.

(2) Projection onto a convex set ($C_P$) defined by the known phase values.

Let $x$ be a signal that is not an element of the set $C_P$. To take the projection of $x$ onto $C_p$, we must choose $x_P$ in $C_P$ so as to minimize

$$\sum_i (x[i]-x_p[i]).(x[i]-x_p[i])^* \tag{A.6}$$

Assuming that the signal samples are independent of each other, each term in the summation must be minimized independently. Let a typical term be $x[j]-x_P[j]$. Let $x_p[j] = A \exp(j\theta)$ and $x[j] = B \exp(j\phi)$. The corresponding term in the summation will be

$$|A.\exp(j\theta)-B.\exp(j\phi)| = \exp(j\theta)|A-B\exp(j(\phi-\theta))|$$

$$= \exp(j\theta)\ [(A-B.\cos(\phi-\theta))^2+(B.\sin(\phi-\theta))^2]$$

It is minimized by taking $A = B.\cos(\phi-\theta)$.

Therefore to take the projection of a signal x onto $C_p$, multiply the magnitude values at all the samples by $\cos(\phi-\theta)$ and replace their phase by the known phase values. Here $\phi$ is the phase value of one of the signal samples and $\theta$ is the known phase value for that sample.

(3) Projection onto the convex set $c_s$ defined by the knowledge of signal samples.

Let A denote the set of points where the signal samples are known and B denote the complement of A. Let x be a signal not in $c_s$. We have to choose an $x_s \in c_s$ such that (A.6) is minimized. We are free to chose the values of $x_s$ over the set of points B only as the values for the points in A are constrained to be the known values. It can be proved that (A.6) is minimized if we retain the values of x at the points in the set B and replace the known values at the set of points in the set A. This is, therefore the procedure to take the projection onto the set $c_s$.

(4) Projection onto the convex set $c_1$ defined by knowledge of 1-bit phase.

Let A denote the set of points where the 1-bit phase of the signal is known. Let B denote the complement of A. Assume that x is a signal that is not an element of the set $c_1$, and therefore, its 1-bit phase over the set A is not as it should have been. This means that at some points in A, the sign of the real part of the signal sample is opposite of the known sign. We have to choose an $x_1 \in C_1$ such that (A.6) is minimized. Each term in the summation has to be minimized independently. The terms corresponding to the points in the set B can be minimized if the values of the samples of $x_1$ are equal to the values of the corresponding samples of x. That leaves us with the points in set A. Assume that the 1-bit phase value of a particular sample $x_1[j]$ differs from that of $x[j]$ for some j in A. Let $x[j] = a + j.b$. It is known that the sign of the real part should have been negative. If the value $c + j.d$ for $x_1[j]$ minimizes the term $|x[j]-x_1[j]|$, it implies that $(a-c)^2 + (b-d)^2$ is minimized. Since c is not greater than zero, the term is minimized by taking $c=0$ and $b=d$. Therefore the following procedure must be followed to take the projection onto the set $C_1$. If the sign of the real part of a sample in the signal is not equal to the known sign of that sample, then we must set the real part for that sample to zero and retain the value of the imaginary part.

(5) Projection onto the convex set C2 defined by knowledge of 2-bit phase.

Following the way the projection was taken onto the set $C_1$, it can be verified that the projection can be taken onto $C_2$ by setting the real part of a signal sample to zero if the sign of the real part is not equal to the known sign for the real part and by setting the imaginary part to zero if the sign of the imaginary part is not equal to the known sign of the imaginary part for that sample.

The procedures for projections onto convex sets presented in this Appendix are independent of the dimension of the signal and can be used for two-dimensional signals also.

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[1] B.Yegnanarayana, C.P.Mariadassou, and Pramod Saini, "Image reconstruction from sensor array data," presented at the Indo-Us workshop, Bangalore, India, Jan. 11-14, 1988 (to appear in the proceedings of the Workshop).

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